SOME ENUMERATIVE DIGRAPHS AND HYPERGRAPHS PROBLEMS

By
MALATI HEGDE

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
AUGUST, 1978

SOME ENUMERATIVE DIGRAPHS AND HYPERGRAPHS PROBLEMS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
MALATI HEGDE

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
AUGUST, 1978

A- 118658

CERTIFICATE

This is to certify that the Ph.D. thesis entitled 'Some Enumerative Digraphs and Hypergraphs Problems' by Mrs. Malati Hegde is a record of bonafide research work carried out by her under my supervision and guidance. She had fulfilled the other requirements for the award of Ph.D. degree. The results embodied in this thesis have not been submitted to any other Institute or University for the award of any degree or diploma.

I.I.T., Kanpur
August 1978

(M.R. Sridharan)
Mathematics Department
I.I.T., Kanpur

ACKNOWLEDGEMENTS

It is with great pleasure that I express my deep sense of gratitude and indebtedness to Dr. M.R. Sridharan who guided me at all stages of my Ph.D. research work. I am grateful to him for his keen interest and constant encouragement.

My thanks are due to Mr. S.K. Shukla for the many useful discussions that we had.

I wish to express my sincere thanks to Mr. Nirmal Roberts for his kind assistance in the use of computer to obtain the values given in the last chapter of the thesis.

I thank Mr. C.M. Abraham for his excellent typing and Mr. Rajeshwari Prasad for cyclostyling this thesis.

Malati Hegde

CONTENTS

INTRODUCTION

Existence problems and enumeration problems are the two types of problems that one comes across in the study of combinatorial mathematics in general and graph theory in particular. It may be observed that enumerative graph theory is establishing itself as a very important branch of graph theory. The very fact that there are three books [24], [28] and [26] and more than 600 references (as listed out by D.M. Jackson) written on this subject stands as a proof for its fast growing importance.

Chronologically, Cayley [4] initiated research in this branch of graph theory. His interest was in the problem of counting of the isomers of saturated hydrocarbons. While editing the mathematical works of Hurwitz, Polya came across a reference to this problem. As he had suggested some methods to problem solvers in [33], he wanted to try these techniques for this problem . The result is the celebrated theorem of Polya [32]. In the meantime Redfield, who worked as a professor of romance languages and later practised as a civil engineer, published an outstanding paper [40], [22] and [38]. It came to limelight much later. Independently Read proved the superposition theorem in [34] and De Bruijn has extended Polya's theorem in [6], [8] and [7]. Harary solved some problems using Polya's theorem [16].

He had suggested many open problems in [17], [18], [23] and [24]. Foulkes [12] and [13] and Sheehan [46] and [47] pointed out the interrelationships between the representation theory of groups and the theorems of Redfield and Polya. Harrison [14] made use of Polya's theorem in the theory of mathematical machines. We refer to [15] for a very interesting discussion of enumerative problems in connection with switching theory or threshold logic. White pointed out in [54] and [55] the relationship between Redfield theorem and multilinear algebra. Williamson [56] and [57] made a similar attempt for operator theory and Debruijn's theorem. Now quite a good number of survey articles are available in [18], [23], [38] and [50]. The use of computers for enumerative problems is discussed in [39]. In [30] and [1] some results are given. It is interesting to note that inspite of the vast number of research papers published already, there are many open problems which are challenging and difficult to solve as given in [24]. The reason is that each problem requires a new technique and a bold approach.

Klarner [27] and Robinson [42] have used the idea of the sum of the cycle indices of the automorphism groups of graphs. Another method is to apply the Burnside's lemma [3] or the weighted version as given in [24]. In this thesis the weighted form is used frequently to solve the problems.

Cayley's identification of saturated hydrocarbons with trees where every vertex has degree 1 or 4 made researchers focus their attention on the problem of counting of graphs and digraphs with their partitions. To mention the works of a few, Parthasarathy [31], Harary and Palmer [21] and Parthasarathy and Sridharan [48] and [53] have found the solutions for some problems of this type. Following their lead, we solve four problems dealing with partitions in the first chapter. The results are presented for the partitions of selfcomplementary oriented graphs, selfconverse digraphs and oriented graphs, and tournaments.

The number of eulerian graphs was obtained by Robinson [41] and the more difficult problem of deriving the formula for the number of eulerian digraphs was solved in [51]. In the second chapter, we give the number of even digraphs. This is followed by the solution of the problem of enumerating mixed digraphs whose complement and converse are isomorphic.

One of the most interesting papers in enumeration is by Read [36]. Attracted by the method used by him, we extend the result for the number of labelled even digraphs in Chapter 3. We also count the numbers of labelled k-coloured and strongly k-coloured hypergraphs and also labelled even hypergraphs.

A topic of current interest to researchers is the study of hypergraphs. A book by Berge [2] on this subject gives detailed results. In [9] also, we find interesting information.

Klarner and deBruijn in [28] and Harary and Frucht in [11] have dealt with this concept. In Chapter 4, we offer solutions for the problems of counting hypergraphs, directed hypergraphs, oriented hypergraphs and selfcomplementary hypergraphs and dihypergraphs.

Enumeration of transitive digraphs is a very difficult problem. It is related to the problem of finding the number of finite topologies. It is also known that the human brain sends the message transitively. Harary et al [10] used computer to obtain the number for labelled transitive digraphs. Other works in this subject include [45] and [29]. In Chapter 5, we enumerate the number of selfcomplementary transitive digraphs and tabulate the values. It is proved that there is only one hamiltonian transitive digraph and the uniquesness of eulerian transitive digraph is established.

Chapter 1

ENUMERATION WITH PARTITIONS

In this chapter we enumerate some classes of digraphs with partitions. The method of enumeration has as its basis a weighted version of Burnside's lemma. This lemma states that the number of equivalence classes induced by a permutation group on a finite set is equal to the average number of elements of the finite set fixed by each element of the group.

Harary and Palmer [19] have counted selfconverse digraphs. Sridharan [49] has obtained the formula for the oriented cases of selfcomplementary, and selfconverse digraphs. The counting of tournaments is due to Davis [5]. In this chapter we enumerate the following classes of digraphs with partitions.

1. Selfcomplementary oriented graphs
2. Selfconverse digraphs
3. Selfconverse oriented graphs
4. Tournaments

1.1 Basic Definition

We start with some basic definitions.

Definition 1.1

A digraph is a pair $D = (V,E)$, where V is a nonempty finite set of elements called vertices and E is a subset of the set $V^{[2]}$ of ordered pairs of distinct elements of V. The elements of E are called edges of the digraph D.

If $e = (a,b)$ is an edge of a digraph then 'a' is adjacent to 'b' and 'b' is adjacent from 'a'.

Definition 1.2

A digraph is an oriented graph if at the most one of (u,v) or (v,u) is an edge for all distinct elements u and v of the vertex set.

Definition 1.3

Mixed digraph is a digraph which contains both ordinary and directed edges .

Definition 1.4

Two digraphs (V,E) and (V,F) are defined to be isomorphic if these exists a permutation π of V such that $\left\{(\pi u,\ \pi v) : (u,v) \in E\right\} = F$.

Definition 1.5

If the p vertices of a digraph D are given distinct labels $v_1, v_2, v_3, \ldots v_p$, then D is a labelled digraph.

Definition 1.6

Two labelled digraphs D_1 and D_2 are considered isomorphic if and only if there exists a 1-1 mapping from $V(D_1)$ onto $V(D_2)$ which preserves adjacency and labelling also.

Definition 1.7

Suppose (V,E) is a digraph, and $v \in V$; then the indegree of v is defined to be the number $\left|\left\{u : (u,v) \in E\right\}\right|$, while the out degree of v is defined to be the number $\left|\left\{u : (v,u) \in E\right\}\right|$.

The total degree of v is the sum of these two numbers.

Definition 1.8

A digraph is defined to be an isograph if every vertex of the digraph has

$$\text{indegree} = \text{outdegree}$$

Definition 1.9

A digraph is an Even digraph if the total degree of every vertex is even number.

Definition 1.10

A tournament is a digraph in which every pair of vertices are joined by exactly one edge.

Definition 1.11

The converse of the digraph (V,E) is the digraph $\left(V, \left\{(v,u) : (u,v) \in E\right\}\right)$.

A self converse digraph is one which is isomorphic to its converse.

Definition 1.12

The complement of the digraph (V,E) is the digraph $(V, \{ (u,v) : u,v \in V,\ u \neq v,\ (u,v) \notin E \})$.

A selfcomplementary digraph is one which is isomorphic to its complement.

Definition 1.13

A digraph is transitive if the presence of two edges (u,v) and (u,w) implies that of (u,w).

Definition 1.14

A digraph is hamiltonian if it contains a spanning directed cycle.

Definition 1.15

An Eulerian digraph is a weakly connected isograph.

Definition 1.16

The ordered partition of a digraph D is a sequence of ordered pairs consisting of the outdegree and the indegree of each vertex in nonincreasing order with respect to outdegree. Here (n_1, n_2) means that the corresponding vertex has outdegree n_1 and indegree n_2. For example, ((2,0), (1,1), (0,2)) means that D has one vertex with degree (2,0), another with (1,1) and the third with (0,2). This is written shortly as $(120, 11^2, 102)$ where abc denotes that there are a vertices with degree (b,c).

Definition 1.17

A hypergraph $H = (V,E) = (E_1, E_2, E_3, \ldots\ldots E_m)$ is a family of subsets E_i of a finite nonempty set of vertices V. The sets E_i's are called edges.

Definition 1.18

A directed hypergraph or dihypergraph $DH = (V,E) = (E_1, E_2, \ldots\ldots E_m)$ is a family of ordered subsets E_i of a set of vertices V. The sets E_i's are called edges.

In both the cases we ignore the edges with cardinality one since we do not want to consider loops.

Definition 1.19

A hypergraph is oriented if at the most one of the $k!$ ordered subsets is an edge for all edges with cardinality k.

Definition 1.20

Two hypergraphs H_1 and H_2 are isomorphic if there exists a one-to-one correspondence α between their vertex sets such that

if $\left\{ v_1, v_2, v_3, v_4, \ldots\ldots v_m \right\}$ is an edge in H_1, then

$\left\{ \alpha v_1, \alpha v_2, \alpha v_3, \ldots\ldots \alpha v_m \right\}$ is an edge in H_2.

Definition 1.21

The complement $\bar{H}$ of a hypergraph H is an hypergraph with the same vertex set as H and $\left\{ v_1, v_2, v_3, \ldots\ldots v_k \right\}$ is an edge in $\bar{H}$ if and only if it is not an edge in H.

A selfcomplementary hypergraph is one which is isomorphic to its complement.

Definition 1.22

The complement $\overline{DH}$ of a directed hypergraph DH is a directed hypergraph with the same vertex set as that of DH and an ordered set $\{v_1, v_2, v_3, \ldots v_k\}$ is an edge in $\overline{DH}$ if and only if $\{v_1, v_2, v_3, \ldots v_k\}$ is not an edge in DH. A self-complementary directed hypergraph is one which is isomorphic to its complement.

Definition 1.23

If p vertices of a hypergraph are given distinct labels $v_1, v_2, v_3, \ldots v_p$, then H is a labelled hypergraph.

Definition 1.24

The degree of a vertex $v \in V$ of a hypergraph is the number of edges E_i of H such that $v \in E_i$.

Definition 1.25

A hypergraph is said to be an Even hypergraph if every vertex has even degree.

Definition 1.26

A colored hypergraph consists of a hypergraph with a vertex set V together with an equivalence relation on V, such that no edge consists of only equivalent vertices, the k-equivalence classes are regarded as k-colors and H is called k-colored.

Definition 1.27

A strong k-colored hypergraph consists of a hypergraph with a vertex set V together with an equivalence relation on V such that no edge contains more than 1 equivalent vertex. The k-equivalence classes are regarded as k-colors and H is strongly k-colored.

Definition 1.28

k-colored (strongly k-colored) hypergraphs are isomorphic if there exists a 1-1 correspondence between their vertex sets which preserves adjacency and colors also.

Definition 1.29

Let G be a permutation group with an object set $V = \{1,2,3, \ldots\ n\}$. An element $g \in G$ is said to have the type $(j_1, j_2, j_3, \ldots\ j_n)$ if there are j_1 cycles of length 1, j_2 cycles of length 2, etc. Then the cycle index of G denoted by Z(G) is a polynomial in the variables $s_1, s_2, s_3, \ldots\ s_n$ defined by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} s_1^{j_1} s_2^{j_2} s_3^{j_3} \ldots\ s_n^{j_n}.$$

We refer to [24] for the definitions and notations not mentioned here.

Weighted version of Burnside's lemma is stated below and for the proof and details we refer to the forthcoming book of Klarner and deBruijn [28].

Lemma : (Weighted version of Burnside's lemma)

Let S denote a finite set, let G denote a finite group, and let χ denote a representation of G by permutations of S. Then S splits into equivalence classes of G under representation χ which are denoted by $S^{(1)}, S^{(2)}, S^{(3)}, \ldots S^{(k)}$. Next w define a weight function w(b) defined on S, and let $W(S^{(i)})$ denote the average weight of the elements of $S^{(i)}$, $i = 1,2,3, \ldots k$; that is,

$$W(S^{(i)}) = \frac{1}{|S^{(i)}|} \sum_{s \in S^{(i)}} w(s) \tag{1.11}$$

with these definitions and notations, it follows that

$$\sum_{i=1}^{k} W(S^{(i)}) = \frac{1}{|G|} \sum_{\gamma \in G} \sum_{\substack{s \in S \\ \chi(\gamma)s=s}} w(s) \tag{1.1.2}$$

In the present context this result reads as follows :

The sum of the weights over the equivalence classes of required classes of digraphs, namely, selfcomplementary oriented graphs (selfconverse digraphs, selfconverse oriented graphs or Tournaments) is

$$\frac{1}{p!} \sum_{\alpha \in S_p} \sum_{\substack{s \in S \\ \chi(\alpha)s=s}} w(s) \tag{1.1.3}$$

where

S = set of all digraphs of the required class (i) or (ii) or (iii) or (iv).

(i) Selfcomplementary oriented graphs

(ii) Selfconverse digraphs

(iii) Selfconverse oriented graphs

(iv) Tournaments

If G is a digraph with j edges and with the partition

(s_1,t_1) (s_2,t_2) (s_3,t_3) ..... (s_p,t_p), then the weight of G is defined as

$$W(G) = x^j \ (s_1,t_1)\ (s_2,t_2)\ (s_3\ t_3) \ldots (s_p,t_p) \qquad (1.1.4)$$

In the case of selfcomplementary digraphs and tournaments the number of edges is constant so the term x^j is dropped finally.

We can write (1.1.3) as

$$\frac{1}{p!} \sum_{\alpha \in S_p} C(\alpha) \qquad (1.1.5)$$

where

$$C(\alpha) = \sum_{\substack{s \in S \\ \mathcal{X}(\alpha)s=s}} w(s) \qquad (1.1.6)$$

1.2 Algorithm :

The procedure is essentially the same as given in [51] and [53]. Let $\alpha \in S_p$ have k cycles $C_1, C_2, C_3, \ldots C_k$. Using the notation from the above section $C(\alpha)$ is obtained as the product of the contributions from the individual cycles and pairs of cycles of α, because such cycles and cycle pairs generate the

edges of a digraph. Actually the weight is a p-tuple but for convenience we make it a k-tuple. The individual cycles C_i affect only the edges generated by the vertices in C_i and the edges generated by the vertex pairs from C_i and C_j. In deriving the general expression for the contributions we give only the i^{th} co-ordinate for the degrees of the vertices in C_i and i^{th} and j^{th} co-ordinates for those of C_i and C_j, even though both of them should be written as k-tuples where the only nonzero entries are the i^{th} co-ordinate and, i^{th} and j^{th} co-ordinates respectively. For the present we simply denote them by (C_i) and (C_i, C_j). These contributions from individual cycles and cycle pairs of $\alpha \in S_p$ are discussed as follows.

Weight contributions :

Set 1 :	(C_1)	(C_1, C_2)	(C_1, C_3)	(C_1, C_4)	......	(C_1, C_k)
Set 2 :		(C_2)	(C_2, C_3)	(C_2, C_4)	......	(C_2, C_k)
Set 3 :			(C_3)	(C_3, C_4)	......	(C_3, C_k)
Set k-1 :					(C_{k-1})	(C_{k-1}, C_k)
Set k :						(C_k)

Now $C(\alpha)$ is obtained by multiplying these factors. This is done as follows. First of all, all subsets within the sets are multiplied. Then the contributions from different sets are multiplied between themselves. Here multiplication

of k-tuples means co-ordinatewise addition. Powers of x are multiplied in the usual way.

1.3 Self Complementary Oriented Graphs

As in [49], any permutation $\alpha \in S_p$ which contributes to selfcomplementary oriented graphs will be of the form $(1^{j_1}\ 2^{j_2}\ 6^{j_6}\ 10^{j_{10}}\ \ldots.)$ where $j_1 = 0$ or 1.

(a) Consider a cycle of length $L = 4r + 2$ where r is an integer $\geqslant 0$. Without loss of generality, we can take this cycle to be of the form (12345 ..... 4r+2). We have two-tuples for every vertex, one for outdegree and other for indegree. We consider the odd positions and even positions occupied by the vertices in every cycle of $\alpha \in S_p$. Under the conditions of selfcomplementary oriented graphs, we have 4r cycles which pair off into converse cycles, and one selfconverse cycle.

Example 1.3.1

α = (123456) gives rise to

((12)(23)(34)(45)(56)(61)) ((21)(32)(43)(54)(65)(16))

((13)(24)(35)(46)(51)(62)) ((31)(42)(53)(64)(15)(26))

((14)(25)(36)(41)(52)(63)).

We write (a,b : c,d) to denote the out degree and indegree of the vertex occupying odd position, outdegree and indegree of vertex occupying even position. That is (a,b) denotes outdegree and indegree of odd position, and (c,d) denotes the the outdegree and indegree of even position.

Thus the contribution from a cycle of length 4r+2 is

$$[(2,0:0,2) + (0,2:2,0)]^r \, [(1,1:1,1) + (1,1:1,1)]^r$$

$$\times \, [(1,0:0,1) + (0,1:1,0)] \qquad (1.3.1)$$

(b) Let C_i and C_j be two cycles of lengths $L_1 = 4a+2$ and $L_2 = 4b+2$. Let m and d be the least commonmultiple and greatest commondivisors of L_1 and L_2 respectively.

Let $\frac{m}{L_1} = p_1$ and $\frac{m}{L_2} = q_1$

In writing the contributions, we adopt the following convention.

$(a_1,b_1 : a_2,b_2; \; c_1,d_1 : c_2,d_2)$ stands for the following :

(a_1,b_1) denotes the outdegrees and the indegrees of the vertices occupying odd position in C_i. (a_2,b_2) gives the outdegrees and the indegrees of the vertices occupying the even position in C_i. (c_1,d_1) means the outdegrees and the indegrees of the vertices occupying odd position in C_j and (c_2,d_2) denotes the outdegrees and the indegrees of the vertices occupying even position in C_j. So the contribution for selfcomplementary oriented graphs can be written as

$$[(p_1,0 : 0,p_1; \; 0,q_1 : q_1,0) + (0,p_1 : p_1,0; \; q_1,0 : 0,q_1)]^{\frac{d}{2}}$$

$$\times [(p_1,0 : 0,p_1; \; q_1,0 : 0,q_1) + (0,p_1 : p_1,0; \; 0,q_1 : q_1,0)]^{\frac{d}{2}} \qquad (1.3.2)$$

Example 1.3.2

(12)(34) gives rise to the following converse pairs.

((13)(24)) ((31)(42))

((14)(23)) ((41)(32))

The contribution can be written as

$[(1,0 : 0,1; 0,1 : 1,0) + (0,1 : 1,0; 1,0 : 0,1)]$

$\times [(1,0 : 0,1; 1,0 : 0,1) + (0,1 : 1,0; 0,1 : 1,0)]$

(c) If C_i is of unit length and C_j is of length $4a+2$ $a \geqslant 0$, then the contribution is

$$[(2a+1,\ 2a+1;\ 0,1 : 1,0) + (2a+1,\ 2a+1;\ 1,0 : 0,1)] \qquad (1.3.3)$$

The number of selfcomplementary oriented graphs of order p, for p = 3,4,5,6 are calculated as given below

For p = 3

The permutation which contributes to the above class is $\alpha_1 = (1)(23)$

Let $C_1 = (1)$ and $C_2 = (23)$

From (1.3.1) and (1.3.3) the contributions can be written as follows :

(C_1, C_2) gives $[(1,1; 0,1 : 1,0) + (1,1; 1,0 : 0,1)]$

(C_2) gives $[(1,0 : 0,1) + (0,1 : 1,0)]$

after multiplication contribution to $C(\alpha_1)$ is

$(1,1;\ 1,1\ :1,1) + (1,1;\ 0,2\ :2,0) + (1,1;\ 2,0\ :0,2)$

$+ (1,1;\ 1,1\ :1,1)$

Therefore the partition of selfcomplementary oriented graphs of order 3 is $((2,0)(1,1)(0,2)) + ((1,1)(1,1)(1,1))$

That is, $(120,11^2,102) + (31^2)$.

$p = 4$

The only permutation which contributes to self-complementary oriented graphs is $\alpha = (12)(34)$

Let $C_1 = (12)$, $C_2 = (34)$

(C_1) gives $\lfloor(1,0\ :0,1) + (0,1\ :1,0)\rfloor$

(C_1,C_2) gives $\lfloor(2,0\ :0,2;\ 1,1\ :1,1) + (1,1\ :1,1;\ 0,2\ :2,0)\rfloor$

(C_2) gives $\lfloor(1,0\ :0,1) + (0,1\ :1,0)\rfloor$

After multiplication, we get the contribution to $C(\alpha)$ as

$(3,0\ :0,3;\ 2,1\ :1,2) + (2,1\ :1,2;\ 1,2\ :2,1)$

$+ (2,1\ :1,2;\ 3,0\ :0,3) + (1,2\ :2,1;\ 2,1\ :1,2)$

$+ (2,1\ :1,2;\ 2,1\ :1,2) + (1,2\ :2,1;\ 1,2\ :2,1)$

$+ (1,2\ :2,1;\ 3,0\ :0,3) + (0,3\ :3,0;\ 2,1\ :1,2)$

$+ (3,0\ :0,3;\ 1,2\ :2,1) + (2,1\ :1,2;\ 0,3\ :3,0)$

$+ (2,1\ :1,2;\ 2,1\ :1,2) + (1,2\ :2,1;\ 1,2\ :2,1)$

$+ (2,1\ :1,2;\ 1,2\ :2,1) + (1,2\ :2,1;\ 0,3\ :3,0)$

$+ (1,2\ :2,1;\ 2,1\ :1,2) + (0,3\ :3,0;\ 1,2\ :2,1)$.

The partition of selfcomplementary oriented graphs of order 4 is (130, 121, 112, 103) + (221, 212).

Similarly, for $p = 5$

$(140,\ 32^2,\ 104) + (52^2) + (140,\ 131,\ 12^2,\ 113,\ 104)$

$+\ 3(131,\ 32^2,\ 113) + 2(231,\ 12^2,\ 213).$

$p = 6$

(341, 314) + 3(332, 323) + 2(241, 132, 123, 214)

+ 4(141, 232, 223, 114) + (150, 232, 223, 105)

+ (150, 141, 132, 123, 114, 105).

1.4 Selfconverse Digraphs :

(a) Consider a cycle of length $L = 2a + 1$. Let this cycle be of the form (1234 .... 2a+1). Under the conditions for selfconverse digraphs as in [19], this cycle induces, 'a' cycles, each of length 4a+2. So the contribution from a cycle of length 2a+1 is

$$[(0,0\ :\ 0,0) + x^{2L}(2,2\ :\ 2,2)]^a \tag{1.4.1}$$

(b) Consider a cycle of length $L = 4a$, a is an integer $\geqslant 0$. This cycle induces 4a-1 cycles each of length 4a. Hence the contribution is

$$[(0,0\ :\ 0,0) + x^L(2,0\ :\ 0,2)]^a\,[(0,0\ :\ 0,0) + x^L(0,2\ :\ 2,0]^a$$
$$\times\,[(0,0\ :\ 0,0) + x^L(1,1\ :\ 1,1)]^{2a-1} \tag{1.4.2}$$

(c) A cycle of length $L = 4a+2$ induces $4a$ cycles, each of length $4a+2$ and 2 cycles of length $2a+1$.

Hence the contribution is

$$[(0,0 : 0,0) + x^L (2,0 : 0,2)]^a [(0,0 : 0,0) + x^L (0,2 : 2,0)]^a$$
$$x [(0,0 : 0,0) + x^L (1,1 : 1,1)]^{2a} [(0,0 : 0,0) + x^{\frac{L}{2}} (1,0 : 0,1)]$$
$$x [(0,0 : 0,0) + x^{\frac{L}{2}} (0,1 : 1,0)] \qquad (1.4.3)$$

(d) Consider two cycles C_i and C_j of lengths L_1 and L_2 where both are not odd. Cycles C_i and C_j induce 2d cycles each of length m. So the contribution is

$$[(0,0 : 0,0; 0,0 : 0,0) + (p_1,0 : 0,p_1; 0,q_1 : q_1,0)x^m]^{\frac{d}{2}}$$
$$x [(0,0 : 0,0; 0,0 : 0,0) + x^m(0,p_1 : p_1,0; q_1,0 : 0,q_1)]^{\frac{d}{2}}$$
$$x [(0,0 : 0,0; 0,0 : 0,0) + x^m (p_1,0 : 0,p_1; q_1,0 : 0,q_1)]^{\frac{d}{2}}$$
$$x [(0,0 : 0,0; 0,0 : 0,0) + x^m (0,p_1 : p_1,0; 0,q_1 : q_1,0)]^{\frac{d}{2}} \qquad (1.4.4)$$

(e) If C_i is a cycle of unit length and C_j is a cycle of even length L, then the contribution is

$$[(0,0 : 0,0) + x^L(\tfrac{L}{2},\tfrac{L}{2}; 0,1 : 1,0)]$$
$$x [(0,0 : 0,0) + x^L(\tfrac{L}{2},\tfrac{L}{2}; 1,0 : 0,1)] \qquad (1.4.5)$$

(f) Let C_i and C_j be two cycles of lengths L_1 and L_2, where L_1 and L_2 are odd. They induce d cycles of length 2m so the contribution is

$$[(0,0 : 0,0; 0,0 : 0,0) + x^{2m} (p_1,p_1 : p_1,p_1; q_1,q_1 : q_1,q_1)]^d \qquad (1.4.6)$$

Example 1.4.1

Here we obtain the number of selfconverse digraphs of order 3 with partitions.

Let $\alpha_1 = (1)(2)(3)$

$\alpha_2 = (1)(23)$

$\alpha_3 = (123)$

For $C(\alpha_1)$, $C_1 = (1) \quad C_2 = (2) \quad C_3 = (3)$

(C_1, C_2) yields $(0,0;\ 0,0) + x^2\ (1,1;\ 1,1)$

(C_1, C_3) yields $(0,0;\ 0,0;\ 0,0) + x^2\ (1,1;\ 0,0;\ 1,1)$

From (C_2, C_3) we get $(0,0;\ 0,0;\ 0,0) + x^2\ (0,0;\ 1,1;\ 1,1)$

After multiplication we get the contribution as

$$(0,0;\ 0,0;\ 0,0) + x^2(1,1;\ 0,0;\ 1,1) + x^2(0,0;\ 1,1;\ 1,1)$$
$$+ x^4(2,2;\ 1,1;\ 1,1) + x^2(0,0;\ 1,1;\ 1,1) + x^4(1,1;\ 1,1;\ 2,2)$$
$$+ x^4(1,1;\ 2,2;\ 1,1) + x^6(2,2;\ 2,2;\ 2,2)$$

and similarly, $C(\alpha_2)$ will be

$$(0,0;\ 0,0) + x(1,0\ :\ 0,1) + x(0,1\ :\ 1,0) + x^2(1,1\ :\ 1,1)$$
$$+ x^2(1,1;\ 1,0\ :\ 0,1) + x^3(1,1;\ 2,0\ :\ 0,2) + x^3(1,1;\ 1,1\ :\ 1,1)$$
$$+ x^4(1,1;\ 2,1\ :\ 1,2) + x^2(1,1;\ 0,1\ :\ 1,0) + x^3(1,1;\ 1,1\ :\ 1,1)$$
$$+ x^3(1,1;\ 0,2\ :\ 2,0) + x^4(1,1;\ 1,2\ :\ 2,1) + x^4(2,2;\ 1,1\ :\ 1,1)$$
$$+ x^5(2,2;\ 2,1\ :\ 1,2) + x^5(2,2;\ 1,2\ :\ 2,1) + x^6(2,2;\ 2,2,\ :\ 2,2)$$

$C(\alpha_3)$ will be

$$[(0,0 : 0,0) + x^6(2,2 : 2,2)]$$

Now using (1.1.5) the number of selfconverse digraphs with. partitions of order 3 is

$$(300) + x(110,\ 101) + x^2(21^2,\ 100)\ x^2(11^2,\ 110,\ 101)$$
$$+ x^3(120,\ 11^2,\ 102) + x^3(31^2) + x^4(12^2,\ 21^2) + x^4(121,\ 11^2,\ 112)$$
$$+ x^5(12^2,\ 121,\ 112) + x^6(32^2).$$

1.5 Selfconverse Oriented Graphs :

We have seen in [49]

(a) Cycles of odd length do not contribute to selfconverse oriented graphs.

(b) Two odd cycles do not contribute to selfconverse oriented graphs.

(c) Consider a cycle of length $L = 4a$ a an integer $\geqslant 0$. This induces one selfconverse cycle and $4a-2$ cycles of length L which pair off into converse cycles. Therefore, the contribution from a cycle of length $L = 4a$ is

$$[(0,0 : 0,0) + x^L(2,0 : 0,2) + x^L(0,2 : 2,0)]^a$$
$$\times [(0,0 : 0,0) + 2x^L(1,1 : 1,1)]^{a-1} \qquad (1.5.1)$$

(d) Consider a cycle of length $L = 4a+2$, a $\geqslant 0$. This induces $4a$ cycles of length L, and 2 cycles of length $2a+1$, where all pair off into converse cycles. So the contribution to selfconverse oriented graphs is

$$[(0,0 : 0,0) + x^{L}(2,0 : 0,2) + x^{L}(0,2 : 2,0)]^{a}$$

$$\times [(0,0 : 0,0) + 2x^{L}(1,1 : 1,1)]^{a}$$

$$\times [(0,0 : 0,0) + x^{2a+1}(1,0 : 0,1) + x^{2a+1}(0,1 : 1,0)] \qquad (1.5.2)$$

(e) Consider cycles C_i and C_j of lengths L_1 and L_2 (not both odd).

These cycles induce 2d cycles, each of length m which pair off into converse cycles. Therefore the contribution is

$$[(0,0 : 0,0;\ 0,0 : 0,0) + x^{m}(p_1,0 : 0,p_1;\ 0,q_1 : q_1,0) + x^{m}(0,p_1 : p_1,0;\ q_1,0 : 0,q_1)]^{\frac{d}{2}}$$

$$\times [(0,0 : 0,0;\ 0,0 : 0,0) + x^{m}(p_1,0 : 0,p_1;\ q_1,0 : 0,q_1) + x^{m}(0,p_1 : p_1,0;\ 0,q_1 : q_1,0)]^{\frac{d}{2}} \qquad (1.5.3)$$

(f) If C_i is of length 1 and C_j is of length L, L is not odd the contribution is

$$[(0,0 : 0,0) + x^{L}(\tfrac{L}{2},\tfrac{L}{2};\ 1,0 : 0,1) + x^{L}(\tfrac{L}{2},\tfrac{L}{2};\ 0,1 : 1,0)] \qquad (1.5.4)$$

Example 1.5.1

We will calculate the number of selfconverse oriented graphs of order 3 with partitions.

The only permutation which will contribute to selfconverse oriented graphs is $\alpha = (1)(23)$

To find $C(\alpha)$, let $C_1 = (1)$ $C_2 = (23)$

(C_1,C_2) gives $(0,0;\ 0,0) + x^2(1,1;\ 1,0 : 0,1)\ x^2(1,1;\ 0,1 : 1,0)$

(C_2) gives $(0,0 : 0,0) + x(1,0 : 0,1) + x(0,1 : 1,0)$.

after multiplication we get

$(0,0 : 0,0) + x(0,0; 1,0 : 0,1) + x(0,0; 0,1 : 1,0) +$
$+ x^2(1,1; 0,1 : 1,0) + x^2(1,1; 1,0 : 0,1) + x^3(1,1; 1,1 : 1,1)$
$+ x^3(1,1; 0,2 : 2,0) + x^3(1,1; 1,1 : 1,1) + x^3(1,1; 2,0 : 0,2).$

So the number of selfconverse oriented graph of order 3 with partitions is calculated using (1.1.5) as

$(30^2) + x(110, 101, 100) + x^2(11^2, 110, 101) + x^3(31^2)$
$+ x^3(120, 11^2, 102)$

The number of selfconverse oriented graph of order 4 with partitions is given below.

$(40^2) + x(110, 101, 20^2) + x^2(210, 201) + x^2(11^2, 110, 101, 100)$
$+ x^3(120, 11^2, 102, 100) + x^3(120, 110, 101, 102)$
$+ x^3(11^2, 110, 101, 100) + x^4(220, 202) + x^4(41^2)$
$+ 2x^4(120, 21^2, 102) + x^5(130, 21^2, 103) + x^5(121, 120, 112, 102)$
$+ 2x^5(121, 21^2, 112) + x^6(130, 121, 112, 103) + x^6(221, 212).$

1.6 Tournaments :

As in [24], the permutations of $\alpha \in S_p$ which contribute to the above class is of the type $(1^{j_1}\, 3^{j_3}\, 5^{j_5}\, \ldots\ldots)$.

(a) Consider a cycle C_i of length $L = 2a+1$.

This cycle induces 2a cycles, each of length L which pair off into converse cycles. Hence the contribution can be written as

$$[(1,1 : 1,1) + (1,1 : 1,1)]^a \qquad (1.6.1)$$

(b) Consider two cycles C_i and C_j of lengths $L_1 = 2a+1$ and $L_2 = 2b+1$. Cycles C_i and C_j induce 2d cycles, each of length m, which pair off into converse cycles. So contribution is

$$[(p_1,0 : p_1,0; 0,q_1 : 0,q_1) + (0,p_1 : 0,p_1; q_1,0 : q_1,0)]^d \qquad (1.6.2)$$

Example 1.6.1 :

The number of tournaments of order 4 with partitions is calculated. The permutations which contribute to the class of tournaments are : $\alpha_1 = (1)(2)(3)(4)$ and $\alpha_2 = (1)(234)$.

(i) To find $C(\alpha_1)$:

Let $C_1 = (1)$, $C_2 = (2)$, $C_3 = (3)$, $C_4 = (4)$.

Using (1.6.2) the different contributions are as follows :

Corresponding to (C_1,C_2) the contributions $[(1,0;0,1)+(0,1;1,0)]$

'' (C_1,C_3) '' $[(1,0;0,0;0,1)+(0,1;0,0;1,0)]$

'' (C_1,C_4) '' $[(1,0;0,0;0,0;0,1)+(0,1;0,0;0,0;1,$

'' (C_2,C_3) '' $[(0,0;1,0;0,1)+(0,0;0,1;1,0)]$

'' (C_3,C_4) '' $[(0,0;0,0;1,0;0,1)+(0,0;0,0;0,1;1,$

(ii) To find $C(\alpha_2)$

Let $C_1 = (1)$, $C_2 = (234)$

(C_1,C_2) gives $[(3,0; 0,1) + (0,3; 1,0)]$

(C_2) gives $[(1,1 : 1,1) + (1,1 : 1,1)]$

After multiplication and using the formula (1.1.5), the number of tournaments of order 4 with partitions is

$$(130,\ 121,\ 112,\ 103) + (130,\ 312) + (321,\ 103) + (221,\ 212).$$

Chapter 2

SOME CLASSES OF DIGRAPHS

In this chapter we count some classes of digraphs. The method adopted is again a weighted version of Burnside's lemma. Here we count the following.

1. Even digraphs
2. Mixed digraphs with complement and converse isomorphic

2.1 Even Digraphs :

Robinson [41] has counted Eulerian graphs by using Burnside's lemma. Here we extend the result by counting even digraphs. As in the case of previous chapter, we make use of the weighted version of Burnside's lemma, where the weight is equal to one.

Theorem 2.1

The number of even digraphs of order p is

$$E_p = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod_k k^{j_k} j_k!} \quad 2^{e(j)} \tag{2.1.1}$$

where

$$e(j) = \sum_k (k-1)\, j_k + \sum_k 2k\binom{j_k}{2} + \sum_{r<t} 2(r,t) j_r j_t + (1-\sum_k j_k) \tag{2.1.2}$$

Proof

In the present context the lemma says that the sum of the weights of the equivalence classes of even digraphs will be equal to

$$\frac{1}{p!}\sum_{\alpha\in S_p} C(\alpha) \quad \text{where} \quad C(\alpha) = \sum_{\substack{s\in S\\ \chi(\alpha)s=s}} 1. \tag{2.1.3}$$

S = set of all even digraphs

χ = representation of S_p by permutations of S.

To find $C(\alpha)$:

(a) First we find the contribution from cycles of α individually. Let Z_k be a cycle of length k. Z_k induces (k-1) cycles each of length k on ordered pairs of vertices. In each of these cycles every vertex of Z_k lies on even number of edges. So in forming even digraphs each may be excluded or included. So the contribution from j_k cycles of length k towards $2^{e(j)}$ will be

$$2^{\sum_k (k-1)j_k} \tag{2.1.4}$$

(b) Now we consider the contributions from different cycles that is from pair of vertices lying on 2 different cycles. Consider two cycles Z_r and Z_t of lengths r and t of α, and 2-subsets which have one vertex in each of these cycles. Select a cycle Z. Let W be a collection of cycles of 2-subsets, obtained by choosing, for each cycle $Z_r \neq Z$, one of cycles of 2-subsets whose vertices are in Z and Z_r. W will have $(\sum j_k - 1)$ cycles. Now we can determine an even digraph

for each selection of these cycles of 2-subsets not in W. This is done by either including or excluding cycles in W as necessary to make each an even digraph. The number of such cycles not in W is

$$\sum_{r<t} 2(r,t)\, j_r\, j_t + \sum_k 2k \binom{j_k}{2} - \left(\sum_k j_k - 1\right) \qquad (2.1.5)$$

The total contribution from $\alpha \in S_p$ to $C(\alpha)$ is $2^{e(j)}$ where

$$e(j) = \sum_k (k-1) j_k + \sum_k 2k\binom{j_k}{2} + \sum_{r<t} 2(r,t) j_r\, j_t - \left(\sum_k j_k - 1\right) \qquad (2.1.6)$$

Therefore using the formula (2.1.3) the number of even digraphs of order p is

$$= \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod_k k^{j_k}\, j_k!}\, 2^{e(j)}$$

where

$$e(j) = \sum_k (k-1) j_k + \sum_k 2k\binom{j_k}{2} + \sum_{r<t} 2(r,t) j_r\, j_t + \left(1 - \sum_k j_k\right) \qquad (2.1.7)$$

Hence the theorem.

The numbers of even digraphs of order p, upto p = 7 are calculated and given below.

p	1	2	3	4	5	6	7
E_p	1	2	6	38	712	50224	13946352

2.2 Mixed Digraphs with Complement and Converse Same :

Harary and Palmer [20] have counted mixed digraphs. Sridharan [52] has counted mixed selfcomplementary and

selfconverse digraphs by using weighted version of Burnside's lemma. Let S = set of all mixed digraphs with complement and converse isomorphic.

χ a representation of S_p by permutations of S . Let x correspond to an oriented edge, and y to a nonoriented edge.

Using weighted version of Burnside's lemma, the sum of the weights of the equivalence classes of required class will be

$$\frac{1}{p!} \sum_{\alpha \in S_p} \sum_{\substack{s \in S \\ \chi(\alpha)s=s}} x^{\lambda} y^{m} \tag{2.2.1}$$

that is

$$\frac{1}{p!} \sum_{\alpha \in S_p} C(\alpha) \tag{2.2.2}$$

where

$$C(\alpha) = \sum_{\substack{s \in S \\ \chi(\alpha)s=s}} x^{\lambda} y^{m} \tag{2.2.3}$$

Theorem 2.2

The formula for number of mixed digraphs for which complement and converse are same is equal to

$$\frac{1}{p!} \sum_{\alpha \in S_p} C(\alpha) \tag{2.2.4}$$

where

$$C(\alpha) = \prod_{k=4a} a_k^{\left(\frac{k-2}{2}\right)j_k} C_k \prod_{k\,odd} b_{2k}^{\left(\frac{k-1}{2}\right)j_k} \prod_{k\,odd} b_{\langle 2,k\rangle}^{k\binom{j_k}{2}}$$

$$\times \prod_{k\ even} b_k^{k\binom{j_k}{2}} \prod_{\substack{q<r \\ \text{both odd}}} b_{\langle 2,\langle q,r\rangle\rangle}^{(q,r)j_q j_r} \prod_{\substack{q<r \\ \text{at least} \\ \text{one not odd}}} b_{\langle q,r\rangle}^{2(q,r)j_r j_q}$$

Proof

To find $C(\alpha)$

The cycle structure of α which contributes to the class of digraphs having same complement and converse is

$$(1^{j_1}\ 3^{j_3}\ 4^{j_4}\ 5^{j_5}\ 7^{j_7}\ 8^{j_8}\ \ldots\ldots.)$$

The contribution can be written as follows :

(i) A cycle of length $k = 4a$ $a > 0$ contributes

$$a_k^{\frac{k-2}{2}} \quad C_k \qquad \text{where} \quad a_k = 2(x^k + y^{\frac{k}{2}}) \text{ and } C_k = 2y^{k/4}$$

(ii) Odd cycle of length k contributes

$$b_{2k}^{\frac{k-1}{2}} \qquad \text{where} \qquad b_{2k} = 2x^k$$

(iii) Two odd cycles of lengths q and r contribute

$$b_{\langle 2, \langle q,r\rangle\rangle}^{(q,r)}$$

(iv) Two cycles where at least one is even, gives rise to

$$a_{\langle q,r\rangle}^{2(q,r)}$$

Now from (i), (ii), (iii) and (iv) we get the formula for the number of mixed digraphs for which complement and converse isomorphic. Hence the theorem.

The generating functions for mixed digraphs having complement and converse same are calculated for p = 3,4,5 and 6 are given below :

$p = 3$	$2x^3$
$p = 4$	$4x^6 + x^4y + y^3$
$p = 5$	$12x^{10} + 4x^4y^3 + 2y^5$
$p = 6$	$56x^{15} + 16x^{13}y + 16x^9y^3$

Chapter 3

LABELLED ENUMERATION

In this chapter, we are interested in labelled enumerations. We count the following.

1. Labelled k-colored hypergraphs
2. Labelled strongly k-colored hypergraphs
3. Labelled connected hypergraphs
4. Labelled even digraphs
5. Labelled even hypergraphs

R.C. Read [35] has counted the number of k-colored graphs. Here we generalize this result by extending Read's method to obtain the formula for number of labelled k-colored and strongly k-colored hypergraphs.

3.1 Labelled k-colored hypergraphs :

Let $p_1, p_2, p_3, p_4, \ldots\ldots\ldots p_k$ be positive integers that form an ordered partition of p, so that

$$\sum_{i=1}^{k} p_i = p \tag{3.1.1}$$

We write $\{p\}$ for an arbitrary solution of (3.1.1).

Theorem 3.1

The number $C_p(k)$ of k-colored labelled hypergraphs of order p is

$$C_p(k) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots . p_k} 2^{\sum_{m=2}^{p} {}^{p}C_m - \sum_{i=1}^{k} \sum_{m=2}^{p_i} \binom{p_i}{m}} \tag{3.1.2}$$

Proof

Let $C_p(k)$ be the number of k-colored labelled hypergraphs of order p. For a particular assignment of k-colors the total number of k-colored labelled hypergraphs is $k!\ C_p(k)$.

Hence we suppose that the k-colors are fixed. Each solution of $\{p\}$ determines a k-part ordered partition of p. We seek the number of labelled hypergraphs with p_i vertices of i^{th} color. The number of ways the labels can be selected for the vertices is the multinomial

$$\binom{p}{p_1, p_2, p_3, \ldots\ldots p_k} \tag{3.1.3}$$

Now the possible number of subsets that contain at least two nonequivalent vertices is

$$\sum_{m=2}^{p} \binom{p}{m} - \sum_{i=1}^{k} \sum_{m=2}^{p_i} \binom{p_i}{m} \tag{3.1.4}$$

Since each of these subsets may or may not be an edge in forming labelled hypergraphs, we have

$$2^{\sum_{m=2}^{p} \binom{p}{m} - \sum_{i=1}^{k} \sum_{m=2}^{p_i} \binom{p_i}{m}} \quad \text{Choices.} \tag{3.1.5}$$

Now from (3.1.3), (3.1.4) and (3.1.5), the formula for number of k-colored labelled hypergraphs of order p takes the form

$$C_p(k) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots p_k} 2^{\sum_{m=2}^{p} \binom{p}{m} - \sum_{i=1}^{k} \sum_{m=2}^{p_i} \binom{p_i}{m}} \quad (3.1.6)$$

and the generating function for the number of k-colored labelled hypergraphs of order p is

$$C_p(x) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots p_k} (1+x)^{\sum_{m=2}^{p} \binom{p}{m} - \sum_{i=1}^{k} \sum_{m=2}^{p_i} \binom{p_i}{m}} \quad (3.1.7)$$

By substituting m = 2 in (3.1.6) and (3.1.7) we get the formula for number of k-colored labelled graphs.

Some values have been calculated using (3.1.6) and are tabulated below.

p/k	1	2	3	4	5
1	1	0	0	0	0
2	1	2	0	0	0
3	1	24	16	0	0
4	1	2048	1536	2048	0
5	1	21135360	293601280	33554432	67108864

3.2 Labelled Strongly k-colored Hypergraphs :

Theorem 3.2

The number $SC_p(k)$ of strongly k-colored labelled hypergraphs of order p is

$$SC_p(k) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots p_k} 2^{\sum p_1 p_2 + \sum p_1 p_2 p_3 + \cdots + \sum p_1 p_2 \cdots p_k} \qquad (3.2.1)$$

Proof :

Let $SC_p(k)$ be the number of strongly k-colored labelled hypergraphs of order p. For a particular assignment of k-colors, the total number of strongly k-colored hypergraphs is $k!\ SC_p(k)$.

Hence we first consider the case where the k-colors are fixed. Each solution of $\{p\}$, determines a k-part ordered partition of p. We count the number of labelled strongly k-colored hypergraphs with p_i vertices of i^{th} color. The number of ways the labels can be selected for the vertices is the multinomial coefficient

$$\binom{p}{p_1, p_2, p_3, \ \ldots\ p_k} \qquad (3.2.2)$$

Now the possible number of subsets such that each edge has no equivalent vertices is

$$\sum p_1 p_2 + \sum p_1 p_2 p_3 + \qquad + \sum p_1 p_2 p_3 \ \cdots\ p_k \qquad (3.2.3)$$

where $\sum p_1 p_2 p_3 \ \ldots\ p_m$ means, summation over all $\binom{k}{m}$ different choices of m sets from k different classes. For example

For $k = 4$

$$\sum p_1 p_2 = p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4$$

$$\sum p_1 p_2 p_3 = p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 \ .$$

Since each of the subsets in (3.2.3) may or may not be an edge in forming labelled hypergraphs, we have

$$2^{\sum p_1p_2 + \sum p_1p_2p_3 + \cdots + \sum p_1p_2p_3 \cdots p_k} \qquad (3.2.4)$$

Choices.

From (3.2.2), (3.2.3) and (3.2.4) the number of strongly k-colored labelled hypergraphs of order p is

$$SC_p(k) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots p_k} 2^{\sum p_1p_2 + \sum p_1p_2p_3 + \cdots + \sum p_1p_2 \cdots p_k} \qquad (3.2.5)$$

and the generating function for the number of labelled strongly k-colored hypergraphs of order p is

$$SC_p(x) = \frac{1}{k!} \sum_{\{p\}} \binom{p}{p_1, p_2, p_3, \ldots p_k} (1+x)^{\sum p_1p_2 + \sum p_1p_2p_3 + \cdots + \sum p_1p_2 \cdots p_k} \qquad (3.2.6)$$

If we retain $\sum p_1p_2$ and ignore the remaining terms in (3.2.5) and (3.2.6), we get the formulas for labelled k-colored graphs. Values upto p = 6 and k = 6 are calculated and are given in Table 1.

Naturally for k = 2 this agrees with the values given in [24].

3.3 Labelled Connected Hypergraphs :

Theorem 3.3

The number of connected hypergraphs is given by

$$C_p = 2^{2^p - p - 1} - \frac{1}{p} \sum_{k=1}^{p-1} k \binom{p}{k} C_k \, 2^{2^{p-k} - (p-k) - 1} \qquad (3.3.1)$$

Table 1

p/k	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2	0	0	0	0
3	1	12	16	0	0	0
4	1	80	768	2048	0	0
5	1	120	71680	2621440	67108844	0
6	1	9152	15769600	23799191040	$30x2^{40}$	2^{57}

Proof :

A rooted hypergraph has one of its vertices,called the root, distinguished from others. The two rooted hypergraphs are isomorphic if there is a one-to-one mapping from a vertex set of one hypergraph to the other which preserves not only adjacency but also the roots. Now we can obtain the recursive formula as below.

The different rooted, labelled hypergraphs are obtained when a labelled hypergraph is rooted at each of its vertices. Hence the number of rooted labelled hypergraphs of order p is pH_p, where H_p is the number of labelled hypergraphs of order p. The number of labelled hypergraph in which the root is in a component of exactly k vertices is

$$kC_k\binom{p}{k}H_{p-k} \qquad (3.3.2)$$

Now summing from $k = 1$ to p

$$pH_p = \sum_{k=1}^{p} k\binom{p}{k} C_k H_{p-k} \tag{3.3.3}$$

The number of labelled hypergraphs of order p without loop is

$$H_p = 2^{2^p - p - 1}$$

So the equation (3.3.3) reduces to

$$p\, 2^{2^p - p - 1} = \sum_{k=1}^{p-1} k\binom{p}{k} C_k H_{p-k} + pC_p.$$

that is,

$$C_p = 2^{2^p - p - 1} - \frac{1}{p} \sum_{k=1}^{p-1} k\binom{p}{k} C_k\, 2^{2^{p-k} - (p-k) - 1}$$

We give values upto $p = 5$ below.

p	1	2	3	4	5
C_p	1	1	12	1816	67071164

3.4 Labelled Even Digraphs :

R.C. Read [36] has enumerated labelled even graphs. Following Read's method here we generalize the result to labelled even digraphs, and labelled even hypergraphs. As a special case we get the formula for labelled even graphs from labelled hypergraphs. The method is to assign the sign to the digraph (hypergraph) by assigning signs to its vertices

and in turn to the edges. That is, the sign of the digraph (hypergraph) will be a function of signs of the vertices. These signs are assigned in such a way that the unwanted digraphs (hypergraphs), cancel out when summed over all possible assignment of signs to the vertices, and possible labelled digraphs (hypergraphs). And the formula for the number of even digraphs (even hypergraphs) is found by considering the sign of the digraph (hypergraph) as a function of signs of edges. Thus the two ways of summing the sign of labelled digraphs (hypergraphs) overaall possible labelled digraphs (hypergraphs) and assignments of signs give the result. First we discuss labelled even digraphs.

Let L be the set of all labelled digraphs with p vertices and q edges. Consider any digraph D in L and arbitrarily assign +1 and -1 to the vertices. Let the sign of an edge be the product of the signs of the end vertices. The sign of the digraph D, denoted by $\sigma(D)$ be the product of the signs of the edges of D. There are 2^p ways in which the signs can be assigned to the labels of a given digraph. There are $\binom{p(p-1)}{q}$ labelled digraphs with p vertices and q edges.

Let α = sum of the indegrees of the vertices with negative sign

β = sum of the outdegrees of the vertices with positive sign.

Therefore,

$$\sigma(D) = (-1)^{\alpha+\beta} = (-1)^{a+b} \tag{3.4.1}$$

where

a = number of edges from positive to negative vertices

b = number of edges from negative to positive vertices

Now we can write

$$\sum_{D \in L} \left\{ \sum_{S} (-1)^{\alpha+\beta} \right\} = \sum_{S} \left\{ \sum_{D \in L} (-1)^{a+b} \right\} \tag{3.4.2}$$

where

S = set of all possible assignments of -1 and +1 to the vertices, and there are 2^p elements in S.

The left hand side of (3.4.2)

If D is an even digraph then for all $v \in V$ net degree of v is even. This means that $\alpha + \beta$ = even if D is an even digraph. Hence

$$\sum_{S} \sigma(D) = 2^p \quad \text{if} \quad D \text{ is even digraph.}$$

If D is not even, at least one vertex v has net degree odd. The sign $\sigma(D)$ is -1 if odd number of odd vertices have negative sign. The allocations in S for which the label of v is positive and for which it is negative is equinumerous. Hence D contributes nothing to L.H.S. of (3.4.2). Hence L.H.S. of (3.4.2) is 2^p times the number of even digraphs in L.

Right hand side of (3.4.2)

Consider an allocation in S for which n vertices are positive and $m = p - n$ are negative. There are $\binom{p}{n}$ such allocations. Consider the contribution to the R.H.S. of (3.4.2) from those digraphs for which $a+b+d = q$ where d = number of edges with positive sign. Observe that d edges can be chosen in $\binom{m(m-1) + n(n-1)}{d}$ different ways,

a edges can be chosen in $\binom{nm}{a}$ different ways, and

b edges can be chosen in $\binom{nm}{a}$ different ways.

Thus the contribution to the R.H.S. of (3.4.2) from those digraphs for which $a+b+d = q$ is

$$\sum_{a+b+d=q} \binom{p}{n}(-1)^{a+b} \binom{nm}{a}\binom{nm}{b} \binom{m(m-1)+n(n-1)}{d} \tag{3.4.3}$$

This is the coefficient of x^q in the series

$$\binom{p}{n} (1-x)^{2mn} (1+x)^{m(m-1)+n(n-1)} \tag{3.4.4}$$

Now summing for $n+m = p$ we obtain a series in which the coefficient of x^q is the R.H.S. of (3.4.2).

$$\sum_{n+m=p} \binom{p}{n}(1-x)^{2mn} (1+x)^{m(m-1)+n(n-1)} \tag{3.4.5}$$

$$= \sum_{n+m=p} \binom{p}{n} (1-x)^{2mn} (1+x)^{p(p-1)-2nm} \tag{3.4.6}$$

Therefore the number of labelled even digraphs

is equal to

$$\frac{1}{2^p}\,(1+x)^{p(p-1)} \sum_{n=0}^{p} \binom{p}{n} \left(\frac{1-x}{1+x}\right)^{2n(p-n)} \tag{3.4.7}$$

The total number of labelled even digraphs of order p is obtained by putting $x = 1$ in (3.4.7). Hence, number of labelled even digraphs of order p

$$= \frac{1}{2^p}\, 2^{p(p-1)}\, 2 = 2^{p(p-1) - p + 1} = 2^{(p-1)^2}$$

Counting series for labelled even digraphs with any number of vertices and edges is given by the formula

$$E(t,x) = \sum_{p=0}^{\infty} \left\{ \frac{1}{2^p}\,(1+x)^{p(p-1)} \sum_{n=0}^{p} \binom{p}{n}\left(\frac{1-x}{1+x}\right)^{2n(p-n)} \right\} \frac{t^p}{p!}$$

where p! being introduced because the vertices are labelled. And the counting series for connected even digraphs is given by the series $\log E(t,x)$, and the number of connected even digraphs on p labelled vertices and k-edges is p! times the coefficient of $t^p x^k$ in this series.

3.5 Labelled Even Hypergraphs :

The hypergraphs that we are considering here are loop free that is the edges E_i with $|E_i| = 1$ are not considered.

As in the section 3.4, the signs are assigned to the vertices. The sign of an edge is the product of the signs of the vertices it has. And sign of the hypergraph is defined as in the case of digraphs. Now we can have the identity

$$\sum_{H \in B} \left\{ \sum_{S} \sigma(H) \right\} = \sum_{S} \left\{ \sum_{H \in B} \sigma(H) \right\} \qquad (3.5.1)$$

where B = set of all labelled hypergraphs of order p,

and $\sigma(H) = (-1)^a = (-1)^b$ (3.5.2)

where

a = sum of the degrees of the negative vertices

b = number of negative edges.

Consider the L.H.S. of (3.5.1).

If H is an even hypergraph then a is even for all possible allocations of signs to its vertices. Hence

$$\sum_{S} (-1)^a = 2^p \text{ if H is an even hypergraph.}$$

and if H is not even then it contributes zero to L.H.S. of (3.5.1). Thus, L.H.S. of (3.5.1) is 2^p times the number of even hypergraphs in B. Now consider the R.H.S. of (3.5.1). Consider an allocation in S for which n vertices are positive and m = p - n are negative. If there are k edges of negative sign they may occur in

$$\binom{\sum_{i=0}^{n} \binom{n}{i} \left({}^mC_3 + {}^mC_5 + \ldots \right) + \sum_{i=1}^{n} \binom{n}{i}\binom{m}{1}}{k} \quad \text{different ways.}$$

And remaining q-k edges can occur in

$$\binom{\sum_{i=2}^{n} \binom{n}{i} + \sum_{i=0}^{n} \binom{n}{i} \left({}^mC_2 + {}^mC_4 + \ldots \right)}{q-k} \quad \text{different ways.}$$

Note that $\binom{n}{i}\binom{n}{j}$ is zero if $i + j > n$

Now summing from $k = 0$ to q we obtain

$$\sum_{k=0}^{q}(-1)^k\left(\sum_{i=0}^{n}\binom{n}{i}({}^mC_3 + {}^mC_5 + \ldots) + \sum_{i=1}^{n}\binom{n}{i}\binom{m}{1}\right)_k$$

$$\times\left(\sum_{i=2}^{n}\binom{n}{i} + \sum_{i=0}^{n}\binom{n}{i}({}^mC_2 + {}^mC_4 + \ldots.)\right)_{q-k} \qquad (3.5.3)$$

as the contribution to R.H.S. of (3.5.1) for each allocation with a given n and m. This is the coefficient of x^q in

$$(1-x)^{\sum_{i=0}^{n}\binom{n}{i}({}^mC_3 + {}^mC_5 + \ldots) + \sum_{i=1}^{n}\binom{n}{i}\binom{m}{1}}\;(1+x)^{\sum_{i=2}^{n}\binom{n}{i} + \sum_{i=0}^{n}\binom{n}{i}({}^mC_2 + {}^mC_4 + \ldots)}$$

and $\binom{n}{i}\binom{n}{j}$ is zero if $i + j > n$ (3.5.4)

Hence R.H.S. of (3.5.1) is the coefficient of x^q in

$$\sum_{n=0}^{p}\binom{p}{n}(1-x)^{\sum_{i=0}^{n}\binom{n}{i}({}^mC_3 + {}^mC_5 + \ldots.) + \sum_{i=1}^{n}\binom{n}{i}\binom{m}{1}}\;(1+x)^{\sum_{i=2}^{n}\binom{n}{i} + \sum_{i=0}^{n}\binom{n}{i}({}^mC_2 + {}^mC_4 + \ldots.)} \qquad (3.5.6)$$

and this coefficient of x^q is 2^p times the number of even hypergraphs in B. In the formula (3.5.6) if we put

$$\binom{n}{i}\binom{m}{j} = 0 \qquad \text{if} \qquad i + j > 2.$$

it reduces to the case of graphs.

Now the formula (3.5.6) is simplified as follows.

$$\sum_{n=0}^{p} \binom{p}{n} (1+x)^{2^n(1+{}^mC_2 + m^mC_4 + \ldots)-n-1}$$

$$(1-x)^{2^n({}^mC_1 + {}^mC_3 + {}^mC_5 + \ldots)-m}$$

$$= \sum_{n=0}^{p} \binom{p}{n} (1+x)^{2^p-p-1} \left(\frac{(1-x)}{(1+x)}\right)^{2^n({}^mC_1 + {}^mC_3 + {}^mC_5 + \ldots)-m} \quad (3.5.7)$$

Therefore the generating function for the number of labelled hypergraphs of order p

$$= \frac{1}{2^p} \sum_{n=0}^{p} \binom{p}{n} (1+x)^{2^p-p-1} \left(\frac{1-x}{1+x}\right)^{2^n({}^mC_1 + {}^mC_3 + {}^mC_5 + \ldots.)-m} \quad (3.5.8)$$

Now the total number of labelled even hypergraphs of order p is obtained by substituting x = 1 in (3.5.8). Number of labelled even hypergraphs of order p

$$= \frac{1}{2^p} 2^{2^p-p-1} = 2^{2^p-2p-1} \qquad p \geqslant 3$$

The counting series for labelled even hypergraphs with any number of vertices and edges will be

$$A(t,x) = \sum_{p=0}^{\infty} \left\{ \frac{1}{2^p} (1+x)^{2^p-p-1} \sum_{n=0}^{p} {}^pC_n \left(\frac{1-x}{1+x}\right)^{2^n({}^mC_1+{}^mC_3+{}^mC_5 + \ldots)} \right\} \frac{t^p}{p!} \quad (3.5.9)$$

and counting series for connected even hypergraphs is given by its formal logarithm that is log A (t,x).

Chapter 4

HYPERGRAPHS

In this chapter we deal with the enumeration of hypergraphs. Following classes of hypergraphs are counted in this chapter.

1. Hypergraphs
2. Directed hypergraphs
3. Oriented hypergraphs
4. Selfcomplementary directed hypergraphs
5. Selfcomplementary hypergraphs

4.1 Hypergraphs

Enumeration of hypergraphs was done by Klaner and deBruijn [9] using a slightly different approach. The method of enumeration used here is based on the r-dimensional version of polya's theorem. We will state the theorem and for details refer to the book by Klarner and deBruijn [28].

Theorem : (r-dimensional version of Polya's theorem)

Let r-denote a natural number and let

$\bar{A} = (A_1, A_2, A_3, \ldots\ldots\ldots A_r)$ and

$\bar{B} = (B_1, B_2, B_3, \ldots\ldots\ldots B_r)$ denote r-tuples of nonempty, finite sets. Let G be a finite group. Let χ_i denote a representation of G by permutations of A_i $i = 1,2,3, \ldots. r$, and let

$$\bar{B}^{\bar{A}} = B_1^{A_1} \times B_2^{A_2} \times B_3^{A_3} \times \ldots\ldots \times B_r^{A_r}$$

Using the representation $\chi_1, \chi_2, \chi_3, \ldots\ldots \chi_r$, a representation ρ of G by permutations of $\bar{B}^{\bar{A}}$ is defined as

$$\rho(\gamma)(f_1,f_2,f_3, \ldots. f_r) = (f_1(\chi_1(\gamma^{-1}), f_2(\chi_2(\gamma^{-1}), \ldots\ldots f_r(\chi_r(\gamma^{-1})) \qquad (4.1.1)$$

for all $(f_1,f_2,f_3, \ldots. f_r) \in \bar{B}^{\bar{A}}$ and all $\gamma \in G$.

Let the equivalence classes of G in $\bar{B}^{\bar{A}}$ be denoted by $S^{(1)}, S^{(2)}, S^{(3)}, \ldots. S^{(n)}$. Let w_i be a weight function on B_i $i = 1,2,3, \ldots. r$ and a weight of

$(f_1,f_2,f_3, \ldots f_r) \in \bar{B}^{\bar{A}}$ is defined as

$$W(f_1,f_2,f_3, \ldots. f_r) = \sum_{i=1}^{r} \prod_{a \in A_i} w_i f(a) \qquad (4.1.2)$$

Let $W(S^{(j)})$ be the common weight of the elements of S_j for $j = 1,2,3, \ldots. n$. Then the sum of the weights of the equivalence classes is

Now the problem reduces to finding a $Z(S_p^{(k)})$. Our aim is to find the contribution to $S_p^{(k)}$ from $\alpha \in S_p$.

(a) Consider a cycle of length r, and k subsets lying on a single cycle.

Example 4.1.1

Consider a cycle of length 6. 4 subsets are permuted in 2 cycles of length 6 and 1 cycle of length 3.

(123456) gives rise to (1234)(2345)(3456)(4561)(5612)(6123)
(1235)(2346)(3451)(4562)(5613)(6124)
(1245)(2356)(3461)

Example 4.1.2

Consider a cycle of length 7.3 subsets are permuted in 5 cycles each of length 7.

(1234567) gives rise to ((123)(234)(345)(456)(567)(671)(712))
((124)(235)(346)(457)(561)(672)(713))
((125) (236)(347)(451)(562)(673)(714))
((126)(237)(341)(452)(563)(674)(715))
((135)(246)(357)(461)(572)(613)(724))

(i) If r and k are relatively prime that is (r,k) = 1, then k-subsets are permuted in $\frac{{}^rC_r}{r}$ cycles each of length r.

Let A(r,k) = No. of cycles of length r permuted by k subsets lying on a single cycle.

The contribution towards $S_p^{(k)}$ from a cycle of length r is

$$a_r^{A(r,k)} \quad \text{where } A(r,k) = \frac{{}^rC_k}{r} \tag{4.1.5}$$

(ii) If r and k are not relatively prime, that is $(r,k) = d$, then the contribution is

$$A(r,k) = \frac{{}^rC_k - \sum_{b/d > 1} A(\frac{r}{b},\frac{k}{b}) \frac{r}{b}}{r} \quad \text{cycles of length } r, \tag{4.1.6}$$

and $A(\frac{r}{b},\frac{k}{b})$ cycles of length $\frac{r}{b}$, where b divides d. So the contribution towards $S_p^{(k)}$ from j_r a cycle of length r is

$$\prod_r \prod_{b/d} a_{\frac{r}{b}}^{A(\frac{r}{b},\frac{k}{b})\, j_r} \tag{4.1.7}$$

where the product is over all divisors of d, and

$$\begin{aligned} A(n_1,n_2) &= \frac{{}^{n_1}C_{n_2}}{n_1} && \text{if } (n_1,n_2) = 1 \\ &= \frac{{}^{n_1}C_{n_2} - \sum_{b/d > 1} A(\frac{n_1}{b},\frac{n_2}{b}) \frac{n_1}{b}}{n_1} && \text{if } (n_1,n_2) = d \end{aligned} \tag{4.1.8}$$

(b) Consider two cycles of lengths r_1 and r_2.

(i) k-subsets such that k_1 from the cycle of length r_1 and k_2 from the cycle of length r_2 contribute

$$\frac{{}^{r_1}C_{k_1} \; {}^{r_2}C_{k_2}}{\langle r_1, r_2\rangle}$$ cycles each of length $\langle r_1,r_2\rangle$ = l.c.m. of r_1 and r_2.

If $(r_1,k_1) = 1$ and $(r_2,k_2) = 1$,

(ii) k-subsets such that k_1-subsets from cycle of length r_1, and k_2 subsets from cycle of length r_2 contribute

$$A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right)\ A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \cdot \frac{\frac{r_1}{b_1}\,\frac{r_2}{b_2}}{\left\langle \frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle} \quad \text{cycles of length} \quad \left\langle \frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle$$

where $(r_1,k_1) = d_1$, $(r_2,k_2) = d_2$ and b_1 and b_2 are divisors of d_1 and d_2 respectively. $A(n_1,n_2)$ is given by the equation (4.1.8).

Example 4.1.3

Suppose that there are 2 cycles each of length 4.

(1) Consider the case where in each cycle obtained from these two one element is from one cycle and three are from the other cycle. Here, $r_1 = r_2 = 4$, $k_1 = 1$, $k_2 = 3$.
$(r_1,k_1) = (r_2,k_2) = 1$.

4-subsets yield

$$A(4,1)\ \ A(4,3)\frac{4\text{x}4}{\langle 4,4\rangle} \quad \text{cycles of length } 4.$$

Now,
$$A(4,1) = \frac{{}^4C_1}{4} = 1 \quad \text{and} \quad A(4,3) = \frac{{}^4C_3}{4} = 1.$$

(1234)(5678) give rise to ((1567)(2678)(3785)(4856))
((2567)(3678)(4785)(1856))
((3567)(4678)(1785)(2856))
((4567)(1678)(2785)(3856))

(2) Consider 4-subsets such that 2-elements from one cycle and 2 from the other

(1234)(5678) give rise to ((1256)(2367)(3478)(4185))

((1267)(2378)(3485)(4186))

((1278)(2385)(3456)(4167))

((1285)(2356)(3467)(4178))

((1257)(2368)(3475)(4186))

((1268)(2375)(2386)(4157))

((1356)(2467)(3178)(4285))

((2456)(3167)(4278)(1385))

((1357)(2468))

((1368)(2475)).

Hence the contribution from j_{r_1} cycles of length r_1 and j_{r_2} cycles of length r_2, and k-subsets lying on different cycles towards $S_p^{(k)}$ is

$$\prod_{r_1 \leqslant r_2}\ \prod_{\{k\}_2}\ \prod_{b_1/d_1}\ \prod_{b_2/d_2} a_{\left\langle \frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle}^{A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \frac{\frac{r_1}{b_1}\times\frac{r_2}{b_2}}{\left\langle \frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle} j_{r_1}\, j_{r_2}} \qquad (4.1.10)$$

where $\{k\}_2$ denote all the solutions of an ordered 2-part of k. i.e. $k_1 + k_2 = k$.

(c) In general consider i cycles of lengths $r_1, r_2, r_3, \ldots r_i$, and also k-subsets such that k_i elements from a cycle of length r_i. Let, $(r_i, k_i) = d_i$. Let b_i's are divisors of d_i's.

Then the contribution to $S_p^{(k)}$ is

$$\prod_{b_1/d_1} \prod_{b_2/d_2} \cdots \prod_{b_i/d_i}$$

$$A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \cdots A\left(\frac{r_i}{b_i},\frac{k_i}{b_i}\right) \frac{\frac{r_1}{b_1}\frac{r_2}{b_2}\cdots\frac{r_i}{b_i}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\cdots\frac{r_i}{b_i}\right\rangle} \qquad (4.1.11)$$

$$\times\ a_{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\ \cdots\ \frac{r_i}{b_i}\right\rangle}$$

Thus the total contribution from j_{r_1} cycles of length r_1, j_{r_2} cycles of length r_2, etc. j_{r_i} cycles of length r_i and k-subsets lying on different cycles, is

$$\prod_{r_1 \leqslant r_2 \cdots \leqslant r_i} \prod_{\{k\}_i} \prod_{b_1/d_1} \prod_{b_2/d_2} \cdots \prod_{b_i/d_i}$$

$$A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \cdots A\left(\frac{r_i}{b_i},\frac{k_i}{b_i}\right) \frac{\frac{r_1}{b_1}\frac{r_2}{b_2}\cdots\frac{r_i}{b_i}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\ \cdots\ \frac{r_i}{b_i}\right\rangle} j_{r_1}\, j_{r_2} \cdots j_{r_i}$$

$$\times\ a_{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\cdots\ \frac{r_i}{b_i}\right\rangle} \qquad (4.1.12)$$

where $\{k\}_i$ = All solutions for $k_1 + k_2 + \cdots + k_i = k$, i.e. an ordered i-part of k.

Thus from (a), (b) and (c) the formula for number of hypergraphs of order p

is equal to

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c}\, j_c!}\prod_{k=2}^{p}\prod_{i=1}^{k}\prod_{r_1\leqslant r_2\leqslant r_3\cdots\leqslant r_i}\prod_{\{k\}_i}$$

$$\times\prod_{b_1/d_1}\prod_{b_2/d_2}\cdots\prod_{b_i/d_i} A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right)\cdots A\left(\frac{r_i}{b_i},\frac{k_i}{b_i}\right)\frac{\frac{r_1}{b_1}\,\frac{r_2}{b_2}\cdots\frac{r_i}{b_i}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\cdots\frac{r_i}{b_i}\right\rangle}\, j_{r_1}\, j_{r_2}\cdots j_{r_i} \qquad (4.1.13)$$

$$\times\, a_{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\frac{r_3}{b_3}\cdots\frac{r_i}{b_i}\right\rangle}$$

where $d_i = (r_i, k_i)$

Product b_i's are over all divisors of d_i

$\{k\}_i$ = all solutions for eqn. $k_1 + k_2 + \ldots + k_i = k$.
an ordered i-part of k.

$a_r = 1 + x^r$

The above formula, for $k = 2$

$i = 1$ then $k_1 = 2$ $(r,2) = 1$ if r is odd
$(r,2) = 2$ if r is even

and $i = 2$ then $k_1 = 1$ $k_2 = 1$ $(r_1,k_1) = 1$
$(r_2,k_2) = 1$

So the formula (4.1.13) becomes

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c} j_c!}\prod_{r_1\ \mathrm{odd}} a_{r_1}^{A(r_1,2)j_{r_1}}\prod_{r_1\ \mathrm{even}}\left(a_{r_1}^{A(r_1,2)}\ a_{\frac{r_1}{2}}^{A(\frac{r_1}{2},1)}\right)^{j_{r_1}}$$

$$\times\prod_{r_1\leqslant r_2} a_{\langle r_1,r_2\rangle}^{A(r_1,1)\ A(r_1,1)\frac{r_1\cdot r_2}{\langle r_1,r_2\rangle} j_{r_1}\ j_{r_2}} \qquad (4.1.14)$$

and

$$\underset{r=\mathrm{odd}}{A(r_1,2)} = \frac{{}^{r_1}C_2}{r_1} = \frac{r_1-1}{2}$$

$$\underset{r=\mathrm{even}}{A(r_1,2)} = \frac{{}^{r_1}C_2 - A(\frac{r_1}{2},1)\cdot\frac{r_1}{2}}{r_1} = \frac{{}^{r_1}C_2 - \frac{r_1}{2}}{r_1} = \frac{r_1-2}{2}$$

So the formula (4.1.14) reduces to

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c}\ j_c!}\prod_{r_1\ \mathrm{odd}} a_{r_1}^{\frac{r_1-1}{2}j_{r_1}}\prod_{r_1\ \mathrm{even}}\left(a_{r_1}^{\frac{r_1-2}{2}}\ a_{\frac{r_1}{2}}\right)^{j_{r_1}}$$

$$\times\prod_{r_1\leqslant r_2} a_{\langle r_1,r_2\rangle}^{(r_1,r_2)j_{r_1}\ j_{r_2}}$$

which is the formula for the number of graphs of order p.

The generating functions for hypergraphs of order p=2,3,4 are as follows

$n = 2 \qquad 1 + x$

$n = 3 \qquad 1 + 2x + 2x^2 + 2x^3 + x^4$

$n = 4 \qquad 1+3x+7x^2+16x^3+28x^4+35x^5+35x^6+28x^7+16x^8+7x^9+3x^{10}+x^{11}$

Numbers of hypergraphs of order p, p = 2,3,4,5 are calculated and given below.

p	2	3	4	5
No of hypergraphs	2	8	180	603712

4.2 Directed Hypergraphs or Dihypergraph

Here as in the case of hypergraphs the method is same. There is a one-to-one between directed hypergraphs and the elements of $\bar{B}^{\bar{A}}$ with

$$B_1 = B_2 = B_3 \ldots\ldots = \{0,1\}$$

and

$$A_k = V^{[k]} = \text{set of all ordered k-subsets of } V.$$

If the weight $w_i(b) = x^b$ is assigned to each $b \in B_i$ $i = 1,2,3, \ldots\ldots n$, then the weight

$w(f) = x^q$ of $f \in \bar{B}^{\bar{A}}$ indicates that f corresponds to a dihypergraph with q edges. Here in the present problem G is the symmetric group,

$A_k = V^{[k]}$ = set of all ordered k-subsets of V. The group $S_p^{[k]}$ is a permutation group induced by S_p which acts on $V^{[k]}$. A permutation $\alpha \in S_p$ induces a permutation α' in $S_p^{[k]}$ such that, for every element

$$(v_1, v_2, v_3, \ldots\ldots v_k) \in V^{[k]}$$

$$\alpha'(v_1, v_2, v_3, \ldots v_k) = (\alpha v_1, \alpha v_2, \alpha v_3, \ldots \alpha v_k)$$

using (4.1.3)

The formula for the number of dihypergraphs of order p is

$$= \frac{1}{p!} \sum_{\alpha \in S_p} \prod_{k=2}^{p} Z(S_p^{[k]};\ 1+x,\ 1+x^2,\ 1+x^3,\ \ldots..) \qquad (4.2.1)$$

Now to find $Z(S_p^{[k]})$

Our aim is to find the contribution to $S_p^{[k]}$ from $\alpha \in S_p$.

(a) Consider a cycle of length r, and k-subsets lying on a single cycle. The contribution is

$$a_r^{\frac{k!\ {}^rC_k}{r}} \qquad (4.2.2)$$

Example 4.2.1

Consider a cycle of length.

(1234) gives ((123)(234)(341)(412))

((132)(243)(314)(421))

((213)(324)(431)(142))

((231)(342)(413)(124))

((312)(423)(134)(241))

((321)(432)(143)(214))

so 3-subsets are permitted in 6 cycles each of length 4.

So the contribution towards $S_p^{[k]}$, from j_r cycles of length r and k-subsets lying on a single cycle is

$$\prod_r a_r^{\frac{k!\ {}^rC_k}{r}\ j_r} \qquad (4.1.4)$$

(b) Consider two cycles of lengths r_1 and r_2 and k-subsets such that, k_1 elements are from cycle of length r_1 and k_2 elements are from cycle of length r_2. There are,

$$\frac{k!\,({}^{r_1}C_{k_1}\ {}^{r_2}C_{k_2})}{\langle r_1, r_2\rangle} \quad \text{cycles each of length } \langle r_1, r_2\rangle .$$

Example 4.1.2

Consider a cycle of length 2 and a cycle of length 4.

(12)(3456) givesrise to

((1345)(2456)(1563)(2634)) ((1354)(2465)(1536)(2643))

((1435)(2546)(1653)(2364)) ((1453)(2564)(1635) (2346))

((1534)(2645)(1356)(2463)) ((1543)(2654)(1365)(2436))

((2345)(1456)(2563)(1634)) ((2345)(1465)(2536)(1643))

((2435)(1546)(2653)(1364)) ((2453)(1564)(2635)(1346))

((2534)(1645)(2356)(1463)) ((2543)(1654)(2365)(1436))

Hence the contribution from j_{r_1} cycles of length r_1 and j_{r_2} cycles of length r_2 and k-subsets lying on different cycles, towards $S_p^{[k]}$ is

$$\prod_{r_1 \leqslant r_2} \prod_{\{k\}_2} a_{\langle r_1, r_2\rangle}^{\frac{k!\ {}^{r_1}C_{k_1}\ {}^{r_2}C_{k_2}}{\langle r_1, r_2\rangle}\, j_{r_1}\, j_{r_2}} \tag{4.2.5}$$

(c) In general if we consider i cycles of lengths $r_1, r_2, r_3, \ldots r_i$ k-subsets such that k_i rlements are from a cycle of length r_i, then the contribution to $S_p^{[k]}$ is

$$k!\,\frac{{}^{r_1}C_{k_1}\;{}^{r_2}C_{k_2}\;\cdots\cdots\;{}^{r_i}C_{k_i}}{\langle r_1, r_2, r_3, \ldots r_i\rangle}\; a_{\langle r_1, r_2, r_3, \;\ldots.\; r_i\rangle} \tag{4.2.6}$$

And the contribution from j_{r_1} cycles of length r_1, j_{r_2} cycles of length r_2 etc. j_{r_i} cycles of length r_i and k-subsets from different cycles towards $S_p^{[k]}$ is

$$\prod_{r_1 \leqslant r_2 \leqslant r_3 \;\ldots \leqslant r_i}\;\prod_{\{k\}_i} a_{\langle r_1, r_2, r_3, \ldots\, r_i\rangle}^{k!\,\frac{{}^{r_1}C_{k_1}\;{}^{r_2}C_{k_2}\;\cdots\;{}^{r_i}C_{k_i}}{\langle r_1, r_2, r_3, \ldots r_i\rangle}\, j_{r_1}\, j_{r_2} \cdots\, j_{r_i}} \tag{4.2.7}$$

Thus the formula for number of directed hypergraphs of order p is

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c}\, j_c!}\;\prod_{k=2}^{p}\;\prod_{i=1}^{k}\;\prod_{r_1 \leqslant r_2 \leqslant r_3 \;\ldots\; \leqslant r_i}\;\prod_{\{k\}_i} \times\; a_{\langle r_1, r_2, r_3, \ldots r_i\rangle}^{k!\,\frac{{}^{r_1}C_{k_1}\;{}^{r_2}C_{k_2}\;\cdots\cdots\;{}^{r_i}C_{k_i}}{\langle r_1, r_2, r_3, \ldots r_i\rangle}\, j_{r_1}\, j_{r_2} \cdots\, j_{r_i}} \tag{4.2.8}$$

where $\qquad a_r = 1 + x^r.$

The above formula for k = 2 is the following :

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c}\, j_c!}\;\prod_r\; a_r^{2!\,\frac{{}^{r}C_2}{r}\, j_r}\;\prod_{r_1 \leqslant r_2} a_{\langle r_1, r_2\rangle}^{2!\,\frac{{}^{r_1}C_1\;{}^{r_2}C_1}{\langle r_1, r_2\rangle}\, j_{r_1}\; j_{r_2}}$$

$$= \frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c}\, j_c!}\;\prod_r\; a_r^{(r-1)j_r}\;\prod_{r_1 \leqslant r_2} a_{\langle r_1, r_2\rangle}^{2(r_1, r_2)\, j_{r_1}\, j_{r_2}} \tag{4.2.9}$$

which is the formula for directed graphs of order p.

The generating function for number of directed hypergraph of order p = 2, 3 are as below.

$p = 2 \qquad 1+x+x^2$

$p = 3 \qquad 1+2x+14x^2+38x^3+90x^4+132x^5+166x^6+132x^7+90x^8$
$\qquad\qquad +38x^9+14x^{10}+2x^{11}+x^{12}$

4.3 Oriented Hypergraphs

As in the earlier chapters, we use the weighted version of Burnside's lemma to calculate the number of oriented hypergraphs. We start with the contribution to dihypergraphs and pick out oriented hypergraphs from these by eliminating the permutations which give rise to non-oriented hypergraphs as was done in [49].

Let $C_k(\alpha)$ = contribution to oriented hypergraphs from $S_p^{[k]}$ (contribution to k-uniform hypergraphs by permutation $\alpha \in S_p$).

Then by Burnside's lemma, the formula for the number of oriented hypergraphs of order p can be written as

$$\frac{1}{p!} \sum_{\alpha \in S_p} \prod_{k=2}^{p} C_k(\alpha)$$

The formula for the number of directed hypergraph of order p is

$$\frac{1}{p!} \sum_{\{j\}} \frac{p!}{\prod_c c^{j_c} j_c!} \prod_{k=2}^{p} \prod_{i=1}^{k} \prod_{r_1 \leqslant r_2 \leqslant r_i} \prod_{\{k\}_i}$$

$$k! \frac{{}^{r_1}C_{k_1}\, {}^{r_2}C_{k_2} \cdots {}^{r_i}C_{k_i}}{\langle r_1, r_2, \ldots r_i \rangle} j_{r_1} j_{r_2} \cdots j_{r_i}$$

$$\times\, a_{\langle r_1, r_2, r_3, \ldots r_i \rangle}$$

(a) Consider a cycle of length r and k-subsets from a single cycle.

Let

$A(r,k)$ = Number of cycles of length r permuted by k subsets from a single cycle of length r.

If $(r,k) = 1$, then $A(r,k) = \dfrac{{}^rC_k}{r}$ (4.3.1)

and if, $(r,k) = d$ then, $A(r,k) = \dfrac{{}^rC_k - \sum\limits_{\substack{b/d \\ >1}} A(\frac{r}{b}, \frac{k}{b})\, \frac{r}{b}}{r}$ (4.3.2)

So the contribution towards oriented hypergraphs from cycle of length r is

$$a_r^{A(r,k)} \quad \text{where} \quad a_r = (1 + k!\, x^r) \tag{4.3.3}$$

Hence, total contribution from j_r cycles of length r with k subsets from a single cycle towards oriented hypergraphs is

$$\prod_r a_r^{A(r,k) j_r} \tag{4.3.4}$$

Example 4.3.1

Consider a cycle of length 5 and 3-subsets

(12345) gives rise to ((123)(234)(345)(451)(512))(other 5 cycles)

((124)(235)(341)(452)(513))(other 5 cycles)

(b) Consider two cycles of length r_1 and r_2 and k-subsets such that k_1 elements are from a cycle of length r_1 and k_2 elements are from a cycle of length r_2.

Let $(r_1,k_1) = d_1$ and $(r_2,k_2) = d_2$.

Contribution to oriented hypergraph is

$$\Lambda\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) \Lambda\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \frac{\frac{r_1}{b_1}\,\frac{r_2}{b_2}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle} \quad \text{cycles each of length } \left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle$$

where b_1, b_2 are divisors of d_1, and d_2 such that

$$\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle = \langle r_1, r_2\rangle$$

So the contribution from j_{r_1} cycles of length r_1 and j_{r_2} cycles of length r_2 can be written awas

$$\prod_{\{k\}_2} \prod_{b_1/d_1} \prod_{b_2/d_2} a_{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle}^{\Lambda\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right) \Lambda\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right) \frac{\frac{r_1}{b_1}\,\frac{r_2}{b_2}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2}\right\rangle} j_{r_1}\, j_{r_2}}$$

where $\Lambda(r,k)$ is as defined in 4.3.1 and 4.3.2 and b_1 and b_2 are divisors of d_1 and d_2 such that $\left\langle\frac{r_1}{b_1}, \frac{r_2}{b_2}\right\rangle = \langle r_1, r_2\rangle$.

(c) In general consider i cycles of lengths $r_1, r_2, r_3, \ldots r_i$ and k subsets such that k_i elements are from a cycle of length r_i.

Let $d_i = (r_i, k_i)$

Contribution towards oriented hypergraphs from k-subsets is

$$= \prod_{b_1/d_1} \prod_{b_2/d_2} \prod_{b_i/d_i} a_{\left\langle \frac{r_1}{b_1}, \frac{r_2}{b_2}, \ldots \frac{r_i}{b_i} \right\rangle}^{\frac{A(\frac{r_1}{b_1},\frac{k_1}{b_1})\, A(\frac{r_2}{b_2},\frac{k_2}{b_2}) \ldots A(\frac{r_i}{b_i},\frac{k_i}{b_i})\, \frac{r_1}{b_1} \frac{r_2}{b_2} \ldots \frac{r_i}{b_i}}{\left\langle \frac{r_1}{b_1}, \frac{r_2}{b_2} \ldots \frac{r_i}{b_i} \right\rangle}}$$

where product is over all divisors b_i of d_i

Such that

$$\left\langle \frac{r_1}{b_1}, \frac{r_2}{b_2}, \frac{r_3}{b_3}, \ldots \frac{r_i}{b_i} \right\rangle = \langle r_1, r_2, r_3, \ldots r_i \rangle$$

and $a_r = (1 + k!\, x^r)$

Thus the contribution to oriented hypergraphs from j_{r_1} cycles of length r_1, j_{r_2} cycles of length r_2 .... etc. j_{r_i} cycles of length r_i, and k-subsets from different cycles is

$$\prod_{\{k\}_i} \prod_{b_1/d_1} \prod_{b_2/d_2} \prod_{b_i/d_i} a_{\left\langle \frac{r_1}{b_1}, \frac{r_2}{b_2}, \ldots \frac{r_i}{b_i} \right\rangle}^{\frac{A(\frac{r_1}{b_1},\frac{k_1}{b_1}) \ldots A(\frac{r_i}{b_i},\frac{k_i}{b_i})\, \frac{r_1}{b_1} \frac{r_2}{b_2} \ldots \frac{r_i}{b_i}}{\left\langle \frac{r_1}{b_1}, \frac{r_2}{b_2}, \ldots \frac{r_i}{b_i} \right\rangle} j_{r_1} j_{r_2} \ldots j_{r_i}}$$

Thus the formula for number of oriented hypergraphs of order p is

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c} j_c!}\prod_{k=2}^{p}\prod_{i=1}^{k}\prod_{r_1\leqslant r_2\leqslant r_i}\prod_{\{k\}_i}\prod_{b_1/d_1}\prod_{b_2/d_2}\cdots\cdots\prod_{b_i/d_i}$$

$$\times\ a_{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\frac{r_3}{b_3},\ \ldots\ \frac{r_i}{b_i}\right\rangle}^{A\left(\frac{r_1}{b_1},\frac{k_1}{b_1}\right)A\left(\frac{r_2}{b_2},\frac{k_2}{b_2}\right)\ \ldots\ A\left(\frac{r_i}{b_i},\frac{k_i}{b_i}\right)\frac{\frac{r_1}{b_1}\frac{r_2}{b_2}\cdots\frac{r_i}{b_i}}{\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\ldots\frac{r_i}{b_i}\right\rangle}j_{r_1}j_{r_2}\cdots j_{r_i}}$$

b_i's are divisors of d_i such that

$$\left\langle\frac{r_1}{b_1},\frac{r_2}{b_2},\ldots\frac{r_i}{b_i}\right\rangle = \langle r_1,r_2,\ \ldots\ r_b\rangle$$

For $k = 2$ the above formula reduces to

$$\frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c} j_c!}\prod_{r,\text{odd}} a_{r_1}^{\frac{{}^{r_1}C_2}{r_1}j_{r_1}}\prod_{r_1\text{ even}} a_{r_1}^{\frac{r_1-2}{2}j_{r_1}}\prod_{r_1\leqslant r_2}$$

$$\times\ a_{\langle r_1,r_2\rangle}^{\frac{r_1 r_2}{\langle r_1,r_2\rangle}j_{r_1}j_{r_2}}$$

$$= \frac{1}{p!}\sum_{\{j\}}\frac{p!}{\prod_c c^{j_c} j_c!}\prod_{r_1\text{ odd}} a_{r_1}^{\frac{r_1-1}{2}}\prod_{r_1\text{ even}} a_r^{\frac{r_1-2}{2}j_{r_1}}\prod_{r_1\leqslant r_2} a_{\langle r_1,r_2\rangle}^{(r_1,r_2)j_{r_1}j_{r_2}}$$

where $a_r = 1 + 2x^r$ which agrees with the result in [49].

The numbers of oriented hypergraphs of order $p = 2,3,4$ are given below

p	2	3	4
No. of oriented hypergraphs	2	49	1886325

4.4 Selfcomplementary Dihypergraphs

In [11] Frucht and Harary have counted generalized selfcomplementary orbits. DeBruijn in [6] has agiven a general method count selfcomplementary structure. Here we count selfcomplementary directed hypergraphs using deBruijn's generalization of Polya's theorem.

Hypergraphs H_1 and H_2 are isomorphic up to complementation if

$$\text{either} \quad H_1 \simeq H_2 \quad \text{or} \quad H_1 \simeq \bar{H}_2$$

Let H_1 and H_2 be two hypergraphs specified by mappings f and $g \in \bar{B}^{\bar{A}}$. If a permutation $(\alpha; (0)(1))$, where $\alpha \in S_p$ sends the mapping f to g then f and g represent isomorphic hypergraphs and if $(\alpha;(01))$ where $\alpha \in S_p$ sends f to g then these two represent complementary hypergraphs.

Now as in [6] we can easily prove that the number of dihypergraph upto complementations is

$$\frac{1}{2} \prod_{k=2}^{p} Z(S_p^{[k]} \; ; \; 2,2,2, \; \ldots\ldots)) \; +$$

$$\prod_{k=2}^{p} Z(S_p^{[k]} \; ; \; 0,2,0,2, \; \ldots\ldots) \qquad (4.4.1)$$

So as in [37] the formula for number of selfcomplementary di-hypergraphs is

$$= \prod_{k=2}^{p} Z(S_p^{[k]};\ 0,2,0,2,\ \ldots) \tag{4.4.2}$$

The cycle index of $S_p^{[k]}$ for any value of p is equal to

$$\frac{1}{p!} \sum_{\{j\}} \frac{p!}{\prod_c c^{j_c} j_c!} \prod_{i=1}^{k} \prod_{r_1 \leqslant r_2 \leqslant r_3 \leqslant \cdots\, r_i} \prod_{\{k\}_i} \times\ a_{\langle r_1, r_2, r_3, \ldots r_i\rangle}^{\frac{k!\ {}^{r_1}C_{k_1}\ {}^{r_2}C_{k_2} \cdots\ {}^{r_i}C_{k_i}}{\langle r_1, r_2, r_3, \ldots r_i\rangle} j_{r_1} j_{r_2} \cdots j_{r_i}} \tag{4.4.3}$$

We can easily see that the odd cycles of length > 1 give rise to cycles of odd length in $S_p^{[k]}$, and also j_1 cannot be more than 1.

Therefore the permutations $\alpha \in S_p$ which contribute to selfcomplementary dihypergraphs are of the type

$$1^{j_1}\ 2^{j_2}\ 4^{j_4}\ \ldots\ldots\ldots\ldots\ \ j_1 = 0 \text{ or } 1.$$

Now putting $a_{\langle r_1, r_2, r_3,\ \ldots\ r_i\rangle} = 2$

we have the formula for the number of selfcomplementary di-hypergraphs of order p as

$$\frac{1}{p!}\sum_{\{J\}}\frac{p!}{\prod_c c^{J_c} J_c!}\prod_{k=2}^{p}\prod_{i=1}^{k}\prod_{r_1\leqslant r_2\ \ldots\leqslant r_i}\ \prod_{\{k\}_i} 2^{\frac{k!\ {}^{r_1}C_{k_1}\ {}^{r_2}C_{k_2}\ \ldots\ {}^{r_i}C_{k_i}}{\langle r_1, r_2, \ldots r_i\rangle} J_{r_1} J_{r_2} \ldots J_{r_i}}$$

$$= \frac{1}{p!}\sum_{\{J\}}\frac{p!}{\prod_c c^{J_c} J_c!}\prod_{k=2}^{p}\prod_{i=1}^{k} 2^{\sum_{r_1\leqslant r_2\leqslant r_i}\sum_{\{k\}_i}\frac{k!\ {}^{r_1}C_{k_1}\ {}^{r_2}C_{k_2}\ \ldots\ {}^{r_i}C_{k_i}}{\langle r_1, r_2, \ldots r_i\rangle} J_{r_1} J_{r_2} \ldots J_{r_i}} \tag{4.4.5}$$

$$= \frac{1}{p!}\sum_{\{J\}}\frac{p!}{\prod_c c^{J_c} J_c!}\prod_{k=2}^{p} 2^{\varphi(\alpha)}$$

where

$$\varphi(\alpha) = \sum_{i=1}^{k}\ \sum_{r_1\leqslant r_2\leqslant r_3\ \leqslant r_i}\ \sum_{\{k\}_i}\frac{k!\ \binom{r_1}{k_1}\binom{r_2}{k_2}\ldots\binom{r_i}{k_i}}{\langle r_1, r_2, r_3\ \ldots r_i\rangle} J_{r_1} J_{r_2} \ldots J_{r_i}$$

The above formula for $k = 2$ reduces to

$$\frac{1}{p!}\sum_{\{J\}}\frac{p!}{\prod_c c^{j_c} J_c!}\ 2^{\varphi(\alpha)}$$

where $\varphi(\alpha) = \sum_{r_1}\frac{2!\ {}^{r_1}C_2}{r_1} J_{r_1} + \sum_{r_1\leqslant r_2}\frac{2!\ r_1\ r_2}{\langle r_1, r_2\rangle} J_{r_1} J_{r_2}$

$$= \sum_{r} r-1\, J_r + \sum_{r_1 \leqslant r_2} 2(r_1, r_2)\, J_{r_1} J_{r_2} \qquad (4.4.7)$$

which is the formula for number of selfcomplementary digraphs as given in [37]

No. of selfcomplementary directed hypergraphs upto p=2,3,4 is as follows

p	2	3	4
No. of self-comple.directed hypergraphs	1	32	134273264

4.5 Selfcomplementary Hypergraphs

As in the case of previous section we make use of deBruijn's generalization of Polya's theorem [6]. A hypergraph which is selfcomplementary should have the following property.

$$p_{C_k} = \text{even for all } k = 2,3,4, \ldots\ldots p-1.$$

We can easily prove that

p_{C_k} = even for all k = 2,3, ... p-1 implies that

$$p = 2^a \qquad \text{for some integer } a \geqslant 2 .$$
$$\text{or } p = 2^a + 1$$

Now let us consider the contribution to selfcomplementary hypergraphs from permutations $\alpha \in S_p$. The requirement is that

$(\alpha;\beta)\ f(d) = \beta\ (f(\alpha(d))$ for all

$d \in V^{(k)}$ = set of all k-subsets of V, $k=2,3, \ldots p-1$

and $\beta = (01)$

We know that for k=2 the odd cycles, and the cycles of length 4b+2 b > 0 donot contribute. So the odd cycles and cycles of length 4b+2 b > 0 do not contribute towards selfcomplementary hypergraphs. Again a permutation α which has a cycle of length $r < p$ does not contribute to self-complementary hypergraphs.

Example 4.5.3

Let $p = 2^3$, and consider $\alpha = (1234)(5678)$.

If α gives a non zero contribution for k = 2,3,4, ...

We should have all cycles of even length in $S_p^{(k)}$ induced by α.

But, when k = 4, say, the permutation (1234)(5678) gives 2 cycles of length 1, namely (1234) and (5678) in $S_p^{(4)}$. So in general if α has a cycle of length $4a < p$, then for $k = 4a$, there is one cycle of length 1. The only permutation which contributes to selfcomplementary hypergraphs of order p is a cycle of length p. Hence selfcomplementary hypergraphs has the vertex set of cardinality $= 2^a$ $a > 2$ (since $|V| = 2^a+1$ = odd). To find the contribution to selfcomplementary hypergraphs from a cycle of length $p = 2^a$, let A(p,k) = cycle of length p permuted by k-subsets lying on a cycle.

$$A(p,k) = \frac{{}^pC_k}{p} \qquad \text{if } (p,k) = 1$$

$$= \frac{{}^pC_k - \sum\limits_{\substack{b/d \\ >1}} A(\frac{p}{b},\frac{k}{b})\ p/b}{p} \qquad \text{if } (p,k) = d \text{ and}$$

b divisors of d.

Now k-subsets are permuted in

$A(\frac{p}{b},\frac{k}{b})$ cycles of length $\frac{p}{b}$, b divisors of d, where d = (p,k). In forming selfcomplementary hypergraphs we have 2 choices for each cycle. Hence the contribution to selfcomplementary hypergraphs of order p from a cycle of length $p = 2^a$

$$= 2^{\sum\limits_{b/d} A(\frac{p}{b},\frac{k}{b})}$$

where $A(n_1,n_2)$ is defined in (4.5.1).

Therefore the number of selfcomplementary hypergraphs of order p

$$= \frac{1}{p!}\,\frac{p!}{p}\ 2^{\sum\limits_{k=2}^{p-1}\sum\limits_{b/d} A(\frac{p}{b},\frac{k}{b})} \tag{4.5.2}$$

where d = (p,k) and the summation is over all divisors of d and $A(n_1,n_2)$ is as given in (4.5.1).

Chapter 5

TRANSITIVE DIGRAPHS

One of the most challenging problems in enumerative graph theory is counting of transitive digraphs. This problem is more interesting because it has some applications in biology. It has been observed that the human brain sends any message transitively. This means that instead of sending a message from 'a' to 'b' and from 'b' to 'c' the brain chooses the route from 'a' to 'c' directly.

Related problems were solved by others. Transitive step type relations were enumerated in [42] and vacuously transitive relations in [44]. In this chapter, we count some classes of transitive digraphs. They are listed as follows

1. Selfcomplementary transitive digraphs
2. Eulerian transitive digraphs
3. Hamiltonian transitive digraphs

5.1 Selfcomplementary Transitive Digraphs :

As in the earlier chapters, we use the weighted version of Burnside's lemma.

To get the contribution to selfcomplementary transitive digraphs, we will start with the contributions to self-complementary digraphs and pick out transitive digraphs from these. Hereafter we write in short s.c.t digraphs to mean selfcomplementary transitive digraphs.

We know from [37] that the contribution to selfcomplementary digraphs come from the permutations of S_p of the type

$$(1^{j_1}\ 2^{j_2}\ 4^{j_4}\ 6^{j_6}\ \ldots\ldots) \quad j_1 = 0 \text{ or } 1$$

If α S_p has k cycles, we denote these by $C_1, C_2, C_3, C_4, \ldots C_k$. $C(\alpha)$ is obtained from the contributions from the individual cycles and the pair of cycles of α, because such cycles and cycle pairs generate the edges of a digraph. We arrange them as follows

$$\begin{array}{lccccc}
 & (C_1) & (C_2) & (C_3), & \ldots\ldots\ldots & (C_k) \\
\text{First row} & & (C_1, C_2) & (C_1, C_3) & \ldots\ldots\ldots & (C_1, C_k) \\
\text{2nd row} & & & (C_2, C_3) & \ldots\ldots\ldots & (C_2, C_k) \\
 & & & & & \\
k\text{-}1^{th} \text{ row} & & & & & (C_{k-1}, C_k)
\end{array}$$

where (C_i) and (C_i, C_j) denote respectively the contributions from the individual cycle C_i and the pair of cycles C_i and C_j.

For each choice of s.c.t. digraphs from individual cycles, the contribution from cycle pairs are written. Now for all

different choices for the first row, we calculate the contribution to s.c.t. digraphs from the remaining rows. And $C(\alpha)$ is obtained by adding all these. The weight contributions are then summed over $\alpha \in S_p$ and divided by $p!$ to get the general formula.

5.1.1 Contribution from individual cycles of $\alpha \in S_p$:

(a) Consider a cycle of length 2

(12) gives (12 + 21)

where (ij) denotes the directed edge (i,j). For selfcomplementary digraphs, we have two possibilities namely

$$f(12) = \begin{cases} 0 \\ 1 \end{cases}$$

and both are transitive. So, the contribution is 2^1.

(b) Cycle of length $r > 2$ contributes 4 to s.c.t. digraphs

Consider a cycle of length $r > 2$.

(1234567 ....r) yields

(12 34 56 ..... r-1 r + 23 45 67 ...... r1)

(13 35 57 ..... r-1 1 + 24 46 68 ...... r2)

(14 36 58 ..... r-12 + 25 47 69 ...... r3)

(15 37 59 ..... r-13 + 26 48 610...... r4)

(1r 32 54 ..... r-1r-2 + 21 43 65 ..... rr-1)

A cycle of length r induces r-1 cycles each of length r, and for selfcomplementary digraphs each cycle has two possibilities. They are

$$f(1i) = \begin{cases} 0 \\ 1 \end{cases}$$

To prove that there are only 4 s.c.t. digraph, we concentrate on the first two cycles.

We start with

$$f(12) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{and} \quad f(13) = \begin{cases} 0 \\ 1 \end{cases}$$

and show that,

$$f(12) = \begin{cases} 0 \\ 1 \end{cases} \text{ implies } f(14) = f(16) = f(18) = \ldots = f(1r) = \begin{cases} 0 \\ 1 \end{cases} \tag{5.1}$$

and

$$f(13) = \begin{cases} 0 \\ 1 \end{cases} \text{ implies } f(15) = f(17) = f(19) = \ldots = f(1r-1) = \begin{cases} 0 \\ 1 \end{cases}$$

when we impose transitivity.

(i) Let $f(13) = 1$

then $f(13) = f(35) = f(57) = \ldots = f(r-1\ 1) = 1$

The existence of 13 and 35 forces the existence of 15 for transitity, and similarly the existence of 13 and 37 demands that $f(17) = 1$ etc.

So, $f(13) = 1$ implies $f(15) = f(17) = f(19) = \ldots = f(1r-1) = 1$.

(ii) Let $f(13) = 0$. Then $f(24) = 1$.

$f(24) = 1$ implies $f(46) = f(68) = \ldots = f(r2) = 1$.

If 24 and 46 occur then 26 should occur. So, $f(26) = 1$.

Similarly, 24 and 410 force the existence of 210 i.e. $f(210) = 1$ etc.

So, if

$$f(13) = 0 \text{ then } f(15) = f(17) = f(19) = \ldots = f(1r-1) = 0 \qquad (5.3)$$

(iii) Suppose $f(12) = 1$. Then whether

$f(13) = 0$ or 1, $f(14) = 1 = f(16) = f(18) \ldots = f(1r)$.

This is because of the following fact

If $f(13) = 1$, then $f(13) = 1 = f(34)$, then $f(14) = 1$ for transitivity.

Also we know that

$f(13) = 1$ implies that $f(15) = f(17) = f(19) = \ldots = f(1r-1) = 1$

In this case

$f(15) = f(56) = 1$ implies $f(16) = 1$, and

$f(17) = f(78) = 1$ means that $f(18) = 1$, etc.

So we get

$f(12) = f(14) = f(16) = \ldots = f(1r) = 1$.

If $f(13) = 0$, then

$f(12) = 1 = f(24)$ implies that $f(14) = 1$

Similarly we get

$f(12) = f(14) = f(16) = f(18) = \ldots = f(1r) = 1$

Hence

$f(12) = 1$ implies that $f(12) = f(14) = f(16) = \ldots = f(1r) = 1$

for both $f(13) = \begin{cases} 0 \\ 1 \end{cases}$

(iv) Let $f(12) = 0$, then $f(23) = 1$.

$f(13) = 0$ or 1, $f(14) = f(16) = f(18) = \ldots = f(1r) = 0$.

This is because if $f(13) = 1$, then $f(23) = f(35) = 1$ which implies that $f(25) = 1$ for transitivity. So $f(14) = 0$.

Similarly we get

$f(16) = f(18) = \ldots = f(1r) = 0$.

If $f(13) = 0$, then $f(24) = 1$.

So we have $f(24) = f(45) = 1$, which implies that $f(25) = 1$ for transitivity. That is $f(14) = 0$. Similarly, we get

$f(16) = f(18) = \ldots = f(1r) = 0$.

Hence,

$f(12) = 0$ implies

$$f(12) = f(14) = f(16) = \ldots = f(1r) = 0 \text{ for both } f(13) = \begin{cases} 0 \\ 1 \end{cases} \tag{5.5}$$

Thus we are left with 4 possibilities namely

$$f(12) = f(14) = f(16) = \ldots = f(1r) = \begin{cases} 0 \\ 1 \end{cases}$$

$$f(13) = f(15) = f(17) = \ldots = f(1r-1) = \begin{cases} 0 \\ 1 \end{cases} \tag{5.6}$$

and these four digraphs can be written as follows

1)	12 34 56 ... r-1r	2) 12 14 16 ...... 1r
	13 35 57 ... r-11	32 24 26 ...... 2r
	14 36 58 ... r-12	42 34 36 ...... 3r
	15 37 59 ... r-13	52 54 46 ...... 4r
	.	.
	.	.
	1r 32 54 ... r-1r-2	r2 r4 r6 ...... r-1r

3) 21 43 65 ..... rr-1
31 53 75 ..... 1r-1
41 63 85 ..... 2r-1
51 83 95 ..... 3r-1
.
.
r1 23 45 ..... r-2r-1

4) 21 41 61 .... r1
23 42 62 .... r2
24 43 63 .... r3
25 45 64 .... r4
.
.
2r 4r 6r rr-1 (5.7)

Now observe that in the first digraph we have all r-1 edges starting with 1, and r-1 edges starting with 3 etc. Similarly we have in the 4th digraph all edges starting with 2,4,6, ... etc. Let us denote these digraphs by a vertical figure with subscript L at the bottom to denote the edges are left fixed by the elements. So the first digraph will be denoted as

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ \cdot \\ \cdot \\ \cdot \\ r-1 \end{bmatrix}_L$$

and 4th digraph is

$$\begin{bmatrix} 2 \\ 6 \\ 8 \\ \cdot \\ \cdot \\ \cdot \\ r \end{bmatrix}_L$$

Similarly the 2nd and 3rd digraphs have all edges ending with 2,4,6, ... r and 1,3,5, ... r-1 respectively. These are denoted by vertical figure with subscript R at the bottom to denote that the edges are right fixed. Therefore,

$$\begin{bmatrix} 2 \\ 4 \\ 6 \\ \cdot \\ \cdot \\ \cdot \\ r \end{bmatrix}_R \quad \text{is the 2nd digraph and}$$

$$\begin{bmatrix} 1 \\ 3 \\ 5 \\ \cdot \\ \cdot \\ \cdot \\ r-1 \end{bmatrix}_R \quad \text{is the 3rd digraph}$$

Example 5.1

Consider a cycle of length 6.

(123456) gives (12 34 56 + 23 45 61)

(13 35 51 + 24 46 62)

(14 36 52 + 25 41 63)

(15 31 53 + 26 42 64)

(16 32 54 + 21 43 65)

the four s.c.t. digraphs are

12 34 56	12 34 56	23 45 61	23 45 61
13 35 51	24 46 62	24 46 62	13 35 51
14 36 52	14 36 52	25 41 63	25 41 63
15 31 53	26 42 64	26 42 64	15 31 53
16 32 53	16 32 54	21 43 65	21 43 65

They are denoted by

$$\begin{bmatrix}1\\3\\5\end{bmatrix}_L, \quad \begin{bmatrix}2\\4\\6\end{bmatrix}_R, \quad \begin{bmatrix}2\\4\\6\end{bmatrix}_L \text{ and } \begin{bmatrix}1\\3\\5\end{bmatrix}_R \quad \text{respectively}$$

We consider two cycles, one of length L_1 and another of length L_2. Call these cycles C_1 and C_2. They give rise to $2(L_1, L_2)$ cycles, each of length $\langle L_1, L_2 \rangle$, where (L_1, L_2) = g.c.d. and $\langle L_1, L_2 \rangle$ = l.c.m. of L_1 and L_2. There are $2^{2(L_1,L_2)}$ choices for the selfcomplementary digraphs. Out of $2^{2(L_1,L_2)}$ choices we have to see how many are transitive for different choices of edges from individual cycles, that is, contribution to (C_1, C_2) for each choice of s.c.t. digraph from cycles C_1 and C_2.

Let $\Lambda_k^{(2)}$ denote the contribution for s.c.t. digraphs from k cycles of length 2, for a particular choice of k edges from individual cycles.

Lemma 5.1

Contribution from k cycles of length 2 towards s.c.t. digraphs is

$$2^k \Lambda_k^{(2)} \tag{5.8}$$

$$\Lambda_k^{(2)} = \sum_{r=0}^{k-1} {}^{k-1}C_r \left\{ \Lambda_{k-1-r}^{(2)} \varphi_1(n) + \Lambda_r^{(2)} \varphi_2(x) - \Lambda_{k-1-r}^{(2)} \Lambda_r^{(2)} \right\} \tag{5.9}$$

where

$$\varphi_1(n) = \sum_{n=0}^{r} {}^{r}C_n \; A_n^{(2)}$$

and

$$\varphi_2(x) = \sum_{x=0}^{k-1-r} {}^{k-1-r}C_x \; A_x^{(2)} \qquad (5.10)$$

Initial values are $A_1^{(2)} = 1$; and $A_2^{(2)} = 4$.

Proof

(a) Contribution from 2 cycles of length 2

Consider 2 cycles (12) and (34). We shall prove that $A_2^{(2)} = 4$. Let $C_1 = (12)$ $C_2 = (34)$

$$\begin{array}{lll} (12)\ (34) \text{ gives} & (12 + 21) & (34 + 43) \\ & (13 + 24) & (31 + 42) \\ & (14 + 23) & (41 + 32) \end{array}$$

Now for a particular choice from individual cycles C_1 and C_2, say for $f(12) = 1 = f(34)$, let us consider s.c.t. digraphs.

(i) If $f(13) = 1$, then occurrence of $f(13) = f(34) = 1$ implies that $f(14) = 1$ for transitivity and if $f(31) = 1$, then $f(31) = f(12) = 1$ implies that $f(32) = 1$. So, we get the edges

$$\begin{array}{cc} 13 & 31 \\ 14 & 32 \end{array} \qquad (5.11)$$

(ii) Let $f(13) = 1$, then $f(14) = 1$.
If $f(31) = 0$ that is, $f(42) = 1$, then also $f(32) = 1$ because of the fact that,

if f(41) = 1, then f(34) = f(41) = 1 implies that f(31) = 1 which is a contradiction. Hence, f(32) = 1. So, we have

$$\begin{array}{cc} 13 & 42 \\ 14 & 32 \end{array} \tag{5.12}$$

(iii) Let f(13) = 0, then f(24) = 1. Occurrence of f(12) = f(24) = 1 implies that f(14) = 1 for transitivity, and if f(31) = 1, then f(31) = f(12) = 1 implies that f(32) = 1. So, we get,

$$\begin{array}{cc} 24 & 31 \\ 14 & 32 \end{array} \tag{5.13}$$

(iv) Suppose f(13) = 0, then f(24) = 1, and if f(31) = 0, then f(42) = 1. Now, f(32) = 1. Otherwise f(41) = 1 and f(34) = f(41) = 1 implies that f(31) = 1 which is not true. Hence, we have

$$\begin{array}{cc} 24 & 42 \\ 14 & 32 \end{array} \tag{5.14}$$

In all we get 4 digraphs only, and they are

$$\begin{array}{cc} 13 & 31 \\ 14 & 32, \end{array} \qquad \begin{array}{cc} 13 & 42 \\ 14 & 32, \end{array} \qquad \begin{array}{cc} 24 & 31 \\ 14 & 32, \end{array} \qquad \begin{array}{cc} 24 & 42 \\ 14 & 32 \end{array} .$$

In the first digraph 1 and 3 are left fixed. In the second digraph 1 is left fixed and 2 is right fixed. In the third digraph 3 is left fixed and 4 is right fixed. In the fourth digraph 2 and 4 are right fixed. These can be denoted as below with the help of a horizontal figure

$[1,3;\ ]$, $[1;\ 2]_{34}$, $[3;\ 4]_{12}$ and $[\ ;\ 2,4]$.

The suffixes show with respect to which the elements are fixed. The elements before semicolon are left fixed and after it are right fixed. In general,

$$[1,\ 3,5,\ldots\ r_1-1,\ r_1+1,\ r_1+3,\ \ldots,\ r_1+r_2-1\ ;\quad] \qquad (5.14a)$$

$$[1,3,5,\ \ldots\ r_1-1\ ;\ 2,4,6,\ \ldots\ r_1]_{r_1+1r_1+2r_1+3,\ldots r_1+r_2} \qquad (5.14b)$$

$$[r_1+1,\ r_1+3,\ r_1+5,\ \ldots;\ r_1+2, r_1+4,\ r_1+6,\ \ldots r_1+r_2]_{123\ldots r_1} \qquad (5.14c)$$

$$[\quad ;\ 2,4,6,\ \ldots\ r_1,\ r_1+2,\ r_1+4,\ \ldots\ r_2] \qquad (5.14d)$$

mean the following digraphs

$$\begin{array}{lllllll}
1r_1+1 & 3r_1+1 & 5r_1+1\ \ldots & r_1-1r_1+1 & r_1+11 & r_1+31\ \ldots & r_1+r_2-11 \\
1r_1+2 & 3r_1+2 & 5r_1+2\ \ldots & r_1-1r_1+2 & r_1+12 & r_1+32\ \ldots & r_1+r_2-12 \\
1r_1+3 & 3r_1+3 & 5r_1+3\ \ldots & r_1-1r_1+3 & r_1+13 & r_1+33\ \ldots & r_1+r_2-13 \\
\cdot & & & & & & \\
\cdot & & & & & & \\
1r_1+r_2 & 3r_1+r_2 & 5r_1+r_2\ \ldots & r_1-1r_1+r_2 & r_1+1r_1 & r_1+3r_1\ \ldots & r_1+r_2-1r_1
\end{array} \qquad (5.14e)$$

$$\begin{array}{llllll}
1r_1+1 & 3r_1+1\ \ldots & r_1-1r_1+1 & r_1+12 & r_1+14\ \ldots & r_1+1r_1 \\
1r_1+2 & 3r_1+2\ \ldots & r_1-1r_1+2 & r_1+22 & r_1+24\ \ldots & r_1+2r_1 \\
1r_1+3 & 3r_1+3\ \ldots & r_1-1r_1+3 & r_1+32 & r_1+34\ \ldots & r_1+3r_1, \\
\cdot & & & & & \\
\cdot & & & & & \\
1r_1+r_2 & 3r_1+r_2\ \ldots & r_1-1r_1+r_2 & r_1+r_22 & r_1+r_24\ \ldots & r_1+r_2r_1
\end{array} \qquad (5.14f)$$

$$\begin{array}{llllllll}
r_1+11 & r_1+31 & \ldots & r_1+r_2-11 & 1r_1+2 & 1r_1+4 & \ldots. & 1r_1+r_2 \\
r_1+12 & r_1+32 & \ldots & r_1+r_2-12 & 2r_1+2 & 2r_1+4 & \ldots. & 2r_1+r_2 \\
r_1+13 & r_1+33 & \ldots & r_1+r_2-13 & 3r_1+2 & 3r_1+4 & \ldots. & 3r_1+r_2 \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
r_1+1r_1 & r_1+3r_1 & \ldots & r_1+r_2-1r_1 & r_1r_1+2 & r_1r_1+4 & \ldots. & r_1r_1+r_2,
\end{array} \qquad (5.14g)$$

and

$$\begin{array}{llllllll}
r_1+12 & r_1+14 & \ldots & r_1+1r_1 & 1r_1+2 & 1r_1+4 & \ldots. & 1r_1+r_2 \\
r_1+22 & r_1+24 & \ldots & r_1+2r_1 & 2r_1+2 & 2r_1+4 & \ldots. & 2r_1+r_2 \\
r_1+32 & r_1+34 & \ldots & r_1+3r_1 & 3r_1+2 & 3r_1+4 & \ldots. & 3r_1+r_2 \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
r_1+r_22 & r_1+r_24 & .. & r_1+r_2r_1 & r_1r_1+2 & r_1r_1+4 & \ldots. & r_1r_1+r_2
\end{array} \qquad (5.14h)$$

respectively.

For example,

$$[1,3,5,7,9\ ;\quad],$$

$$[1,3,5\ ;\ 2,4,6]_{78910},$$

$$[7,9\ ;\ 8,10]_{123456},$$

and $$[\quad ;\ 2,4,6,8,10]$$

mean the following digraphs

$$\begin{array}{ccccc}
17 & 37 & 57 & 71 & 91 \\
18 & 38 & 58 & 72 & 92 \\
19 & 39 & 59 & 73 & 93 \\
110 & 310 & 510 & 74 & 94 \\
 & & & 75 & 95 \\
 & & & 76 & 96 ,
\end{array} \qquad
\begin{array}{cccccc}
17 & 37 & 57 & 72 & 74 & 76 \\
18 & 38 & 58 & 82 & 84 & 86 \\
19 & 39 & 59 & 92 & 94 & 96 \\
110 & 310 & 510 & 102 & 104 & 106 ,
\end{array}$$

71	91	18	110	and	72	74	76	18	110
72	92	28	210		82	84	86	28	210
73	93	38	310		92	94	96	38	310
74	94	48	410		102	104	106	48	410
75	95	58	510					58	510
76	96	68	610					68	610

We have $A_2^{(2)} = 4$.

We have also seen that $A_1^{(2)} = 1$. We define $A_0^{(2)} = 1$. And for each choice from individual cycles the contribution towards s.c.t. digraphs is same, and there are 2^2 choices from individual cycles C_1 and C_2. So the contribution from 2 cycles of length 2 is

$$2^2 \, A_2^{(2)}.$$

(b) Contribution from k cycles of length 2, $k \geqslant 3$

Consider k cycles of length 2.

Say $C_1 = (12)$, $C_2 = (34)$, $C_3 = (56)$, ... $C_k = (2k\text{-}1 2k)$

Let the term box mean the contribution towards s.c.t. digraphs from a pair of cycles.

Now,

(12) (34) (56) ...... (2k-1 2k) gives

(12+21) (34+43) ....(2k-12k+2k2k-1)

$$\begin{array}{llll}(C_1,C_2), & (C_1,C_3) & \ldots\ldots\ldots\ldots & (C_1,C_k) \\ & (C_2,C_3), & \ldots\ldots\ldots\ldots & (C_2,C_k) \\ & & (C_3,C_4)\ , & (C_3,C_k) \\ & & & (C_{k-1},C_k)\end{array}$$

For a particular choice from individual cycles, say, 12,34,56, ..., 2k-12k, we have 4 choices for each of the above boxes.

This can be represented diagramatically as given below :

$\boxed{....}$ $\boxed{....}$ $\boxed{....}$ $\boxed{....}$ 1st row

$\boxed{....}$ $\boxed{....}$ $\boxed{....}$ 2nd row

$\boxed{....}$ k-1th row

where in each box there are 4 possibilities for s.c.t. diagraphs.

Now consider 2 boxes (C_1, C_i) and (C_1, C_j) from the first row.

The choice of one element each from these two boxes forces the selection of the suitable elements from the third box (C_i, C_j).

(C_1, C_i) has four elements. They are

$[1, i \, ; \,]$, $[1 \, ; \, 2]_{1i+1}$, $[i \, ; \, i+1]_{12}$ and $[\, ; \, 2, i+1]$

and similarly

(C_1, C_j) has

$[1, j \, ; \,]$, $[1 \, ; \, 2]_{jj+1}$, $[j \, ; \, j+1]_{12}$ and $[\, ; \, 2, j+1]$

Now for all different choices of elements in (C_1, C_i) and (C_1, C_j) let us see what happens to the box (C_i, C_j).

1. Consider the choice $[1,i\ ;\ ]$ and $[\ ;\ 2,j+1]$ from (C_1,C_i) and (C_1,C_j)

$$[1,i\ ;\ ]\ [\ ;2,j+1] \quad \text{means}$$

$$\begin{array}{cccc} 1i & i1 & 2\,j{+}1 & j{+}1\,2 \\ 1\,i{+}1 & i2 & 1\,j{+}1 & j2 \\ & & \otimes & \otimes \\ & & i\,j{+}1 & j\,i{+}1 \end{array}$$

Now we have 2 choices ij or $i{+}1\ j{+}1$ for $\otimes$
But both give nontransitive, since
if we choose ij then $1i$ and ij together implies that $1j$ should occur but $1j$ is not present and if we choose $i{+}1\ j{+}1$ then $i{+}1\ j{+}1$ and $j{+}1\ 2$ together forces the the presence of $i{+}1\ 2$ which is not present. Hence this combination gives non-transitive digraphs.

2. Consider the two elements $[1\ ;\ 2]_{i\,i+1}$ and $[1\ ;\ 2]_{j\,j+1}$ from (C_1,C_i) and (C_1,C_j)

$$[1;2]_{i\,i+1} \quad [1\ ;\ 2]_{j\,j+1} \quad \text{means}$$

$$\begin{array}{cccc} 1i & i{+}1\,2 & 1j & j{+}1\ 2 \\ 1\,i{+}1 & i2 & 1\,j{+}1 & j2. \end{array}$$

Here we can easily verify that all the 4 choices for the box (C_i,C_j) give transitive edges.

3. Similarly, the combination
$[i\ ;\ i+1]_{12}\ [j\ ;\ j+1]_{12}$ admits all the 4 choices for the box (C_i,C_j).

4. We can easily verify that all other combinations give only 1 possibility for the box (C_i, C_j).

For example, consider

$[1,i\ ;\ ]$ and $[1\ ;\ 2]_{jj+1}$ from (C_1, C_i) and (C_1, C_j) respectively.

$[1,i\ ;\ ]\ [1\ ;\ 2]_{jj+1}$ means

1i i1 1j j+12

1i+1 i2 1j+1 j2

occurrence of i1 and 1j+1 implies that i j+1 occurs occurrence of i1 and 1j implies that ij occurs j+1 i+1 also occurs, otherwise we have the choice ji but, occurrence of ji and i1 implies that j1 should occur which is not present. So, the choice for the box (C_i, C_j) will be

ij j+1 i+1

ij+1 ji+1 i.e. $[i\ ;\ i+1]_{jj+1}$.

To find formula for $A_k^{(2)}$

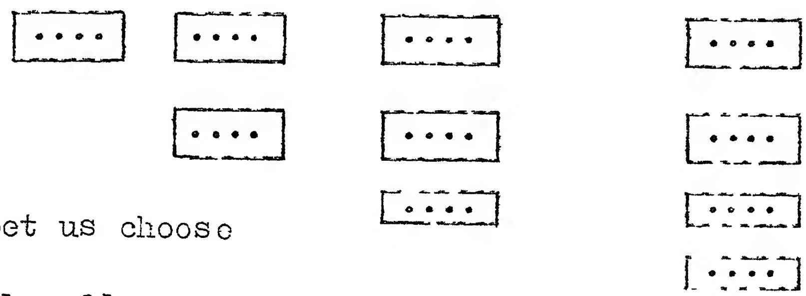
Choose one element each from two boxes in the first row which combine with 4 elements in the third box, in some other suitable row to yield 4 s.c.t. edges. This may happen for some pairs of boxes in the first row. However, the final number of s.c.t. digraphs is not simply a product of these. This is because that among themselves these edges may not satisfy the transitivity.

For example,

consider $\alpha = (12)(34)(56)(78)(910)$

where $C_1 = (12)$, $C_2 = (34)$, $C_3 = (56)$, $C_4 = (78)$, $C_5 = (910)$.

Suppose that we choose 12,34,56,78,910 as the contribution from individual cycles for s.c.t. edges.



Let us choose

$[1\ ;\ 2]_{34}$, $[1\ ;\ 2]_{56}$, $[1\ ;\ 2]_{78}$ and $[9\ ;\ 10]_{12}$ from the first row.

Considering two boxes at a time from the first row we see that the last column contributes only one choice for s.c.t. edges. And the boxes (C_2, C_3), (C_2, C_4) and (C_3, C_4), have 4 choices for s.c.t. edges for the choice of 34,56,78. Now we have 4 elements from each of these boxes but all 4^3 s.c.t. edges are not transitive. The number of choices for these three boxes consistent with transitivity is $A_3^{(2)}$.

For the general case k-1 boxes in the first row are divided into different parts as discussed below

Let r boxes in the first row be of the type

$$[1, i\ ;\ ] \quad \text{or} \quad [1\ ;\ 2]_{j\,j+1}$$

and remaining k-1-r boxes be of the type

$$[\,i;\ i+1]_{12} \quad \text{or} \quad [\ ;\ 2,\ j+1]$$

Now there are three cases

Case 1

If at least one box among the r, in the 1st row is of the type $[1, i\ ;\ ]$, then all k-1-r boxes should be of the type

$$[i\ ;\ i+1]_{12}$$

and say n of the r are of the type $[1\ ;\ 2]_{j\,j+1}$ and r-n are of the type $[1\ ,\ i\ ;\ ]$. So, in all we have

k-1-r boxes are of the type $[i;\ i+1]_{12}$

n boxes are of the type $[1\ ;\ 2]_{j\,j+1}$

r-n boxes are of the type $[1,\ i\ ;\ ]$

So that total number of s.c.t. edges for this particular choice of the 1st row is

$$\Lambda_n^{(2)}\ \Lambda_{k-1-r}^{(2)}$$

Here n varies from 0 to r-1

So, we get

$$\sum_{n=0}^{r-1}\ {}^{r-1}C_n\ \Lambda_n^{(2)}\ \Lambda_{k-1-r}^{(2)} \quad \text{choices} \tag{5.15}$$

Case 2

If at least one box among k-1-r, in the first row is of the type $[\ ;\ 2,\ m+1]$, then all r boxes should be of the type

$[1 ; 2]_{jj+1}$. Say x of the k-1-r are of the type $[1 ; i+1]_{12}$ and k-1-r-x are of the type $[; 2, n+1]$. So in all in the first row we have

k-1-r-x boxes of the type $[; 2, n+1]$

x boxes of the type $[1 ; i+1]_{12}$

r boxes of the type $[1 ; 2]_{jj+1}$

hence we have

$\Lambda_x^{(2)} \Lambda_r^{(2)}$ s.c.t. edges for this particular choice of x.

x varies from 0 to k-1-r-1. So, we have totally

$$\sum_{x=0}^{k-r-2} {}^{k-r-2}C_x \; \Lambda_x^{(2)} \Lambda_r^{(2)} \quad \text{choices} \tag{5.16}$$

Case 3

When no box in the first row is of the type

$[1, i ;]$ or $[; 2, n+1]$

That is, all the r boxes are of the type $[1 ; 2]_{jj+1}$ and all the k-1-r boxes are of the type $[1 ; i+1]_{12}$.

So we have,

$$\Lambda_r^{(2)} \Lambda_{k-1-r}^{(2)} \quad \text{choices} \tag{5.17}$$

Now adding (5.15), (5.16) and (5.17) and varying r from 0 to k-1 we get the total contribution for a particular choice of edges from individual cycles, that is, $\Lambda_k^{(2)}$ as

$$A_k^{(2)} = \sum_{r=0}^{k-1} {}^{k-1}C_r \left\{ \sum_{n=0}^{r-1} {}^{r-1}C_n \, A_n^{(2)} \, A_{k-1-r}^{(2)} + \sum_{x=0}^{k-r-2} {}^{k-r-2}C_x \, A_x^{(2)} A_r^{(2)} + A_{k-1-r}^{(2)} \, A_r^{(2)} \right\} \qquad (5.18)$$

Substituting $\varphi_1(n) = \sum_{n=0}^{r} {}^{r}C_n \, A_n^{(2)}$

and $\varphi_2(x) = \sum_{x=0}^{k-1-r} {}^{k-1-r}C_x \, A_x^{(2)}$

In (5.18) we get the formula for $A_k^{(2)}$ as (5.9).

And the total contribution from k cycles of length 2 is equal to

$$2^k \, A_k^{(2)} \qquad (5.19)$$

Hence the lemma 5.1.

Lemma 5.2

For the contribution from pairs of cycles of length $r > 2$ the length r is immaterial.

Proof

For the contribution from pairs of cycles of length $r > 2$. We will show that the contribution is independent of the length r. In otherwords, k, cycles of length $r_1 > 2$ and k_2 cycles of length $r_2 > 2$ contribute the same as $k_1 + k_2 = k$ cycles of length 4. For this we will consider two cycles, one of length $r1 > 2$ and another of length $r_2 > 2$ and show that the contribution is independent of lengths of the cycles.

Let $C_1 = (123456 \;\ldots.\; r_1)$ and

$$C_2 = (r_1+1 \;\; r_1+2 \;\; r_1+3 \;\; r_1+4 \;\ldots.\; r_1+r_2)$$

We have seen that there are 4 choices from cycles C_1, and C_2 for s.c.t. digraphs. Now for the choice from individual cycles there are 3 different cases to be considered.

Case 1 Left fixed edges from both cycles

Case 2 Left fixed edges from one individual cycle and right fixed edges from the other

Case 3 Right fixed edges from both the cycles.

Case 1 Left fixed edges from both the cycles

Say, we have chosen $\begin{bmatrix} 1 \\ 3 \\ 5 \\ \cdot \\ \cdot \\ r_1-1 \end{bmatrix}_L$ and $\begin{bmatrix} r_1+1 \\ r_1+3 \\ r_1+5 \\ \cdot \\ \cdot \\ r_1+r_2-1 \end{bmatrix}_L$

Now we shall prove that there are only 3 different choices for the box (C_1, C_2), namely

$$[1,3,5, \;\ldots\ldots\; r_1-1,\; r_1+1,\; r_1+3, \;\ldots\; r_1+r_2-1 \;; \qquad] \tag{5.20}$$

$$[1,3,5, \;\ldots\ldots\; r_1-1;\; 2,4,6, \;\ldots\ldots\ldots\; r_1]_{r_1+1, r_1+2 \ldots r_1+r_2} \tag{5.21}$$

$$[r_1+1,\; r_1+3,\; r_1+5, \;..\; r_1+r_2-1;\; r_1+2,\; r_1+4, \;..\; r_1+r_2]_{1234..r_1} \tag{5.22}$$

$(12345\ldots r_1)(r_1+1 \;\; r_1+2 \;\; r_1+3 \;\ldots\; r_1+r_2)$ gives

$(1r_1+1\ \ 3r_1+3\ \dots + 2r_1+2\ \ 4r_1+4\dots)(r_1+11\ r_1+33\ \dots + r_1+22\ r_1+44.)$

$(1r_1+2\ 3r_1+4\ \dots + 2r_1+3\ 4r_1+5\ \dots)\ (r_1+21\ r_1+43\ \dots + r_1+32\ r_1+54\ \dots)$

$(1r_1+3\ 3r_1+5\ \dots + 2r_1+4\ 4r_1+6\dots)(r_1+31\ r_1+53\ \dots + r_1+42\ r_1+64\ \dots)$

.

.

To show that there are only 3 choices we concentrate on the first row.

(1a) Start with

$f(1r_1+1) = 1$, then the presence of $\begin{bmatrix} r_1+1 \\ r_1+3 \\ . \\ . \\ r_1+r_2-1 \end{bmatrix}_L$ means

we have all edges

$r_1+1\ r_1+2,\ r_1+1\ r_1+3,\ r_1+1\ r_1+4,\ \dots\ r_1+1\ r_1+r_2.$

So $f(1r_1+1) = 1 = f(r_1+1\ r_1+i)$ for $i = 2,3,4, \dots r_2$ implies that

$$f(1\ r_1+i) = 1 \quad \text{for } i = 2,3,4, \dots r_2 \tag{5.23}$$

Now if $f(r_1+1\ 1) = 1$, then the presence of $\begin{bmatrix} 1 \\ 3 \\ 5 \\ . \\ . \\ r_1-1 \end{bmatrix}_L$

demands the existence of the edges (r_1+1i) for $i = 1,3,5,,\dots r_1-1$.

So, $f(r_1+1\ i) = 1$ for $i = 1,3,5, \dots r_1-1$ (5.24)

(5.23) and (5.24) give

$[1,3,5, \dots, r_1-1, r_1+1, r_1+3, \dots, r_1+r_2-1\ ;\ \dots]$ (5.25)

(1b) If $f(1\ r_1+1) = 1$ then

$f(1\ r_1+i) = 1$ for all $i = 1,2,3,\ldots\ r_2$ (5.20)

If $f(r_1+1\ 1) = 0$, then $f(r_1+2\ 2) = 1$

and also all

$$f(r_1+i\ 1) = 0 \quad \text{for all } i$$

This is because of the fact that

if $f(r_1+i\ 1) = 1$, then considering the edge $r_1+1\ r_1+i$

from $\begin{bmatrix} r_1+1 \\ r_1+3 \\ \cdot \\ \cdot \\ \cdot \\ r_1+r_2-1 \end{bmatrix}_L$, we have $f(r_1+1\ 1) = 1$ which is a contradiction.

Hence, we have

$$f(r_1+i\ 1) = 0 \quad \text{for all } i = 2,3,4,\ldots\ r_2.$$

So, $f(r_1+i+1\ 2) = 1$ for all $i = 0,1,2,\ldots\ r_2-1$ (5.27)

From (5.26) and (5.27) we get

$[1,3,5,\ldots, r_1-1\ ;\ 2,4,6,8,\ldots\ r_1]_{r_1+1}\ r_1+2\ r_1+3 \ldots\ r_1+r_2.$

(1c) If $f(1\ r_1+1) = 0$, then, $f(2\ r_1+2) = 1$,

and also $f(1\ r_1+i) = 0$ for all i odd.

This is because

$f(1\ r_1+i) = 1$ and considering the edge $(r_1+i\ r_1+1)$

from $\begin{bmatrix} r_1+1 \\ r_1+3 \\ \cdot \\ \cdot \\ r_1+r_2-1 \end{bmatrix}_L$ we should have $f(1r_1+1) = 1$ which is a contradiction.

Now $f(1r_1+i) = 0$ for all $i = 1,3,5, \ldots\ldots r_2-1$ means that $f(2\ r_1+i+1) = 1$ for all $i = 1,3, \ldots r_2-1$

and the presence of $\begin{bmatrix} 1 \\ 3 \\ 5 \\ \cdot \\ \cdot \\ r_1-1 \end{bmatrix}_L$ implies that

all $f(1\ r_1+i+1) = 1$ for $i = 1,3,5, \ldots r_2-1$.

Thus we have

$$f(1\ r_1+i) = 0 \quad \text{for} \quad i = 1,3,5, \ldots r_2-1$$

$$= 1 \quad \text{for} \quad i = 2,4,6, \ldots r_2. \qquad (5.28)$$

If $f(r_1+11) = 1$, then the presence of

$\begin{bmatrix} 1 \\ 3 \\ 5 \\ \cdot \\ \cdot \\ r_1-1 \end{bmatrix}_L$ implies that

all $f(r_1+i\ 1) = 1$ for $i = 1,2,3, \ldots\ldots r_2$ (5.29)

So, we get

$[r_1+1,\ r_1+3,\ r_1+5,\ \ldots,\ r_1+r_2-1;\ r_1+2,\ r_1+4,\ \ldots\ r_1+r_2]_{1234\ldots r_1}$

from (5.28) and (5.29).

(1d) We know that

$f(1\ r_1+1) = 0$ implies that

$$f(1\ r_1+i) = 0 \quad \text{for} \quad i = 1,3,5, \ldots r_2-1$$

$$= 1 \quad \text{for} \quad i = 2,4,6,, \ldots r_2.$$

and

$f(r_1+1\ 1) = o$ implies that $f(r_1+2\ 2) = 1$ and we have the edge $2\ r_1+4$.

So, $f(r_1+2\ r_1+4) = 1$. But in $\begin{bmatrix} r_1+1 \\ r_1+3 \\ \cdot \\ \cdot \\ r_1+r_2-1 \end{bmatrix}_L$,

this edge does not occur.

Hence, this possibility is ruled out. So, in all we get only 3 different choices for the box (C_1, C_2) namely

$$[1,3,5, \ldots, r_1-1, r_1+1, r_1+3, \ldots\ldots, r_1+r_2-1 ;\quad] \tag{5.30}$$

$$[1,3,5, \ldots, r_1-1; 2,4,6, \ldots r_1]_{r_1+1\ r_1+2\ \ldots\ r_1+r_2} \tag{5.31}$$

$$[r_1+1, r_1+3, \ldots r_1+r_2-1; r_1+2, r_1+4, \ldots r_1+r_2]_{1234\ldots r_1} \tag{5.32}$$

Case 2

For the 2nd choice that is left fixed edges from one cycle and right fixed edges from other, we have 2 choices for (C_1, C_2) namely,

$$[1,3,5,7, \ldots r_1-1; 2,4,6,8, \ldots r_1]_{r_1+1\ r_1+2\ r_1+3\ \ldots r_1+r_2} \tag{5.33}$$

$$[r_1+1, r_1+3, \ldots r_1+r_2-1; r_1+2, r_1+4, r_1+6, \ldots r_1+r_2]_{1234\ldots\ r_1} \tag{5.34}$$

Proof is similar to the case 1.

Case 3

For the 3rd choice, that is, right fixed edges from both

the cycles, there are 3 choices for (C_1, C_2). They are

$$[1,3,5, \ldots, r_1-1;\ 2,4,6, \ldots r_1]\ r_1+1\ r_1+2\ r_1+3 \ldots r_1+r_2 \qquad (5.35)$$

$$[r_1+1, r_1+3, r_1+5, \ldots r_1+r_2-1;\ r_1+2,\ r_1+4, \ldots r_1+r_2]\ 12 \ldots r_1 \qquad (5.36)$$

$$[\ \ ;\ 2,4,6, \ldots, r_1,\ r_1+2,\ r_1+4, \ldots r_1+r_2] \qquad (5.37)$$

This can be proved as in the case 1.

Thus the contribution is independent of lengths r_1 and r_2. Hence the lemma 5.2.

Example 5.2

Consider 2 cycles one of length 4 and other of length 8.

Let $C_1 = (1234)$

$C_2 = (56789101112)$

(1234)(56789101112) gives

(15 37 19 311 + 26 48 210 412) (51 73 91 113 + 62 84 102 124)
(16 38110 312 + 27 49 211 45) (61 83101 123 + 72 94 112 54)
(17 39111 35 + 28410 212 46) (71 93111 53 + 82104 122 64)
(18310112 36 + 29411 25 47) (81103121 63 + 92114 52 74)

(a) Let us choose $\begin{bmatrix}1\\3\end{bmatrix}_L$ and $\begin{bmatrix}5\\7\\9\\11\end{bmatrix}_L$ from cycles C_1 and C_2. That
Le
is, left fixed edges from both the cycles C_1 and C_2.

(a1) Start with $f(15) = 1$.

In $\begin{bmatrix}5\\7\\9\\11\end{bmatrix}_L$ we have all the edges 57,58,59,510,511, 512.

which implies that

$$f(16) = f(17) = f(18) = f(19) = f(10) = f(11) = f(12) = 1 \tag{5.38}$$

for transitivity.

Let $f(51) = 1$. The presence of $[\frac{1}{3}]_L$ means that we have edges 12,13,14,

which implies that

$$f(52) = f(53) = f(54) = 1 \text{ for transitivity.} \tag{5.39}$$

So we have

$$[1,3,5,7,9,11\ ;] \tag{5.40}$$

from (5.38) and (5.39)

(aii) We know that $f(15) = 1$ implies that

$$f(16) = f(17) = f(18) = f(19) = f(110) = f(111) = f(112) = 1$$

If $f(51) = 0$, then $f(62) = 1$ (5.41)

and all $f(61) = f(71) = f(81) = f(91) = f(101) = f(111) = f(121) = 0$

because of the fact that

$f(61) = 1 = f(56)$ implies that $f(51) = 1$, which is a contradiction.

This means that we have

$$f(51) = f(61) = \ldots\ldots = f(121) = 0$$

and

$$f(62) = f(72) = \ldots\ldots\ldots = f(52) = 1 \tag{5.42}$$

from (5.41) and (5.42) we have

$$[1,3;\ 2,4]_{56789101112} \tag{5.43}$$

(aiii) If $f(15) = 0$, then $f(26) = 1$.

We also should have

$$f(15) = f(17) = f(19) = f(111) = 0 \tag{5.44}$$

because, $f(17) = f(75) = 1$ implies that $f(15) = 1$ which is a contradiction,

which means that

$$f(26) = f(28) = f(210) = f(212) = 1.$$

And the presence of $\begin{bmatrix}1\\3\end{bmatrix}_L$ implies that

$$f(16) = f(18) = f(110) = f(112) = 1 \tag{5.45}$$

If $f(51) = 1$, then we know that

$$f(61) = f(71) = \ldots\ldots = f(121) = 1 \tag{5.46}$$

So, from (5.44), (5.45) and (5.46) we get

$$[5,7,9,11\ ;\ 6,8,10,12]_{1234} \tag{5.47}$$

(aiv) We know that

$f(15) = o$ implies that

$$f(17) = f(19) = f(111) = 0.$$

and $\quad f(16) = f(18) = f(110) = f(112) = 1.$

and $\quad f(51) = 0$ implies that $f(62) = 1$.

Now we have the edge 28 and 62, so $f(68) = 1$. But in

$\begin{bmatrix}5\\7\\9\\11\end{bmatrix}_L$ the edge 68 does not occur.

Hence this possibility is ruled out. So, in all for (C_1, C_2) we have only 3 choices. Namely,

$[1,3\ ;\ 2,4]_{56789101112}$

$[1,3,5,7,9,11\ ;\ ]$

$[5,7,9,11\ ;\ 6,8,10,12]_{1234}$.

Let

$A^{(4)}_{k(r)}$ denote the contribution to s.c.t. digraphs from k cycles of length 4 for a particular choice of k-r left fixed edges and r right fixed edges from individual cycles.

Lemma 5.3

Contribution from k cycles of length 4 towards s.c.t. digraphs is

$$2^k \sum_{r=0}^{k} {}^kC_r\ A^{(4)}_{k(r)} \tag{5.48}$$

and $A^{(4)}_{k(r)}$ is given by

$$A^{(4)}_{k(r)} = \sum_{n=0}^{r} {}^rC_n \sum_{t=0}^{k-1-r} {}^{k-1-r}C_t \sum_{m=0}^{t} {}^tC_m\ \varphi(t,m,n) \tag{5.49}$$

where

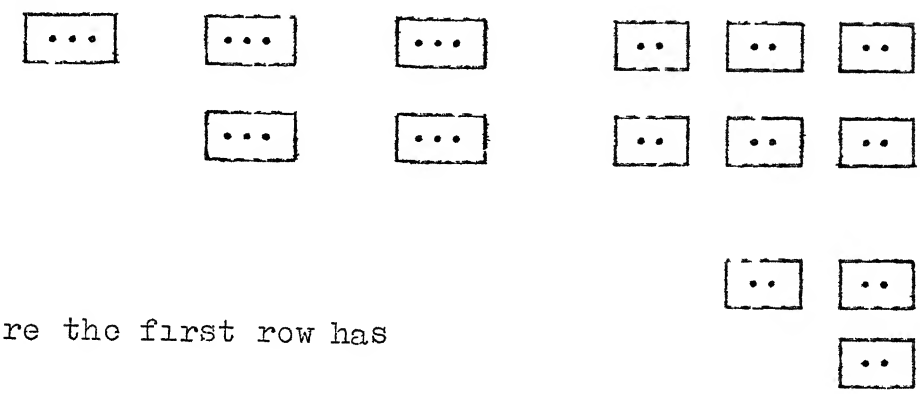
$$\varphi(t,m,n) = A^{(4)}_{k-1-t-n(r-n)}\ A^{(4)}_{m+n(n)} \tag{5.50}$$

Proof

The procedure is same as in the case of cycles of length 2.

Let the cycles be $C_1, C_2, C_3, \ldots C_k$.

Suppose that from cycles $C_1, C_2, C_3, \ldots C_{k-r}$ we have chosen left fixed edges and from $C_{k-r+1}, C_{k-r+2}, \ldots, C_k$ we have the right fixed edges. The situation now can be thought of as follows



where the first row has

$3^{k-1-r}\ 2^r$ possibilities.

For each of the k-1-r box we have 3 choices, namely,

$$[1,3,i,i+2\ ;\ ] + [1,3\ ;\ 2,4]_{ii+1i+2i+3}$$
$$+ [i,i+2\ ;\ i+1;\ i+3]_{1234}$$

and for each of the r boxes we have 2 possibilities, namely

$$[1,3\ ;\ 2,4]_{ii+1i+2i+3} + [i,i+2\ ;\ i+1,i+3]_{1234}$$

Let us consider a general situation for the first row where the k-1 boxes are divided into many parts as discussed below.

Suppose that among, k-1-r,

t are of the type $[1,3,i,i+2\ ;\ ]$ or $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$

and k-r-1-t are of the type $[i,i+2\ ;\ i+1,\ i+3]_{1234}$

and among r

n are of the type $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$

and r-n are of the type $[i,i+2\ ;\ i+1,\ i+3]_{1234}$

Let m be of the type $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$.

So in all we have in the first row

m of the type $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$

t-m of the type $[1,3,i,i+2\ ;\ ]$

k-r-1-t of the type $[i,i+2\ ;\ i+1,i+3]_{1234}$

n of the type $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$

and r-n of the type $[i,i+2\ ;\ i+1,\ i+3]_{1234}$

That is,

m+n of the type $[1,3\ ;\ 2,4]_{ii+1i+2i+3}$

k-1-n-t of the type $[i,i+2\ ;\ i+1,\ i+3]_{1234}$

and t-m of the type $[1,3,i,i+2\ ;\ ]$

Now considering 2 boxes at a time say

(C_1, C_i) and (C_1, C_j), the combinations which give more than one possibilities for s.c.t. edges are

$[1,3\ ;\ 2,4]_{i\,i+1\,i+2\,i+3}$ and $[1,3\ ;\ 2,4]_{j\,j+1\,j+2\,j+3}$

$[i,i+2\ ;\ i+1,\ i+3]_{1234}$ and $[j,j+2\ ;\ j+1,\ j+3]_{1234}$

And the number of possibilities for the third box is either

$A^{(4)}_{2(0)}$ $A^{(4)}_{2(1)}$ or $A^{(4)}_{2(2)}$ depending upon the choice of C_i and C_j.

Totally for this particular t,n and m we have

$A^{(4)}_{k-1-t-n(r-n)}\ A^{(4)}_{m+n(n)}$ choices where

n varies from 0 to r

m varies from 0 to t

t varies from 0 to k-1-r.

So,

$$A^{(4)}_{k(r)} = \sum_{n=0}^{r} {}^{r}C_{n} \sum_{t=0}^{k-1-r} {}^{k-1-r}C_{t} \sum_{m=0}^{t} {}^{t}C_{m}\ A^{(4)}_{k-1-t-n(r-n)}\ A^{(4)}_{m+n(n)} \qquad (5.51)$$

and substituting,

$$\varphi(t,m,n) = A^{(4)}_{k-1-t-n(r-n)}\ A^{(4)}_{m+n(n)}$$

We have

$$A^{(4)}_{k(r)} = \sum_{r=0}^{n} {}^{n}C_{r} \sum_{t=0}^{k-1-r} {}^{k-1-r}C_{t} \sum_{m=0}^{t} {}^{t}C_{m}\ \varphi(t,m,n) \qquad (5.52)$$

and total contribution for s.c.t. digraphs from k cycles of length 4 is

$$2^{k} \sum_{r=0}^{k} {}^{k}C_{r}\ A^{(4)}_{k(r)}\ . \qquad \text{Hence the lemma 5.3.} \qquad (5.53)$$

Let $A^{(2,4)}_{k_1,k_2}(r)$ denote the contribution from k_1 cycles of length 2 and k_2 cycles of length 4, for a particular choice of k_1 edges from 2-cycles and r right fixed and k_2-r left fixed edges from 4-cycles.

Lemma 5.4

The contribution from k_1 cycles of length 2 and k_2 cycles of length 4 towards s.c.t. digraphs is

$$2^{k_1+k_2} \sum_{r=0}^{k_2} {}^{k_2}C_r \; A^{(2,4)}_{k_1,k_2}(r) \tag{2.54}$$

where

$A^{(2,4)}_{k_1,k_2}(r)$ is given by

$$A^{(2,4)}_{k_1,k_2}(r) = \sum_{y=0}^{k_1-1} {}^{k_1-1}C_y \sum_{x=0}^{k_2-r} {}^{k_2-r}C_x \sum_{s=0}^{r} {}^{r}C_s$$

$$\left\{ a(x,y,s)\; b(m,n,s) + C(y,x,s)\; d(t,L,s) - a(x,y,s)\; C(y,x,s) \right\} \tag{5.55}$$

where

$$a(x,y,s) = A^{(2,4)}_{k_1-1-y,\; k_2-x-r}(r-s) \tag{5.56}$$

$$b(m,n,s) = \sum_{n=0}^{y} {}^{y}C_n \sum_{m=0}^{x} {}^{x}C_n \;\; A^{(2,4)}_{n,m+s}(s) \tag{5.57}$$

$$C(y,x,s) = A^{(2,4)}_{y,x+s}(s) \tag{5.58}$$

$$d(t,L,s) = \sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L \; N^{(2,4)}_{k_1-1-y,\,k_2-x-s+L(r-s)}$$

Proof

The procedure is exactly the same as in the case of cycles of length 2.

Consider a cycle of length 2 and a cycle of length 4. Let $C_1 = (12)$ $C_2 = (3456)$

$(12)(3456)$ gives $(12 + 21)$ $\left(\left[{3 \atop 5}\right]_L + \left[{3 \atop 5}\right]_R + \left[{4 \atop 6}\right]_L + \left[{4 \atop 6}\right]_R\right)$

$(13\ 15 + 24\ 26)\ (31\ 51 + 42\ 62)$

$(14\ 16 + 25\ 23)\ (41\ 61 + 52\ 32)$

Case 1

Suppose that we have chosen

12, $\left[{3 \atop 5}\right]_L$ (there are 2^2 such choices).

For the box (C_1, C_2) we have 3 different choices. They are

$[1,3,5\ ;\ ]$

$[1\ ;\ 2]_{3456}$

$[3,5\ ;\ 4,6]_{12}$

Proof is similar as in the case of 2-cycles.

Case 2

Consider now the edges

12 and $\left[{4 \atop 6}\right]_R$

For the box (C_1, C_2), we have 3 choices.

They are

$$[\;;\,2,4,6]$$

$$[1\;;\,2]_{3456}$$

$$[3,5\;;\,4,6]_{12}$$

With the above notation

$$A^{(2,4)}_{1,1(0)} = 3 \quad \text{and} \quad A^{(2,4)}_{1,1(1)} = 3.$$

So, the total contribution from 2 cycles one of length 2 and one of length 4 is

$$2^2\,(A^{(2,4)}_{1,1(0)} + A^{(2,4)}_{1,1(1)}) = 24.$$

Now, consider k_1 cycles of length 2 and k_2 cycles of length 4.

Choose k_1 edges from 2-cycles, and r right fixed and k_2 r left fixed edges from 4-cycles.

Now consider a general situation for the first row k_1+k_2-1 boxes are divided into many parts as below.

Suppose y of the k_1-1 are of the type

$$[1,i\;;\;]\ \text{or}\ [1\;;\,2]_{ii+1},$$

and k_1-1-y are of the type $[1\;;\,i+1]_{12}$ or $[\;;\,2,\,i+1]$.

And out of k_2-r

x are of the type $[1,i,i+2\;;\;]$ or $[1\;;\,2]_{ii+1i+2i+3}$

and k_2-r-x are of the type $[i,i+2;\,i+1,i+3]_{12}$

and out of r say

s of them are of the type $[1\;;\,2]_{ii+1i+2i+3}$

and r-s are of the type $[1,i+2\ ;\ i+1,i+3]_{12}$ or $[\ ;\ 2,i+1,i+3]$

Now there are 2 cases to discuss.

Case 1

If at least one box in the first row is of the form

$[1,i\ ;\ ]$

then all k_1-1-y boxes should be of the type $[1;\ ,i+1]_{12}$

and all r-s boxes are of the form $[1,i+2\ ;\ i+1,i+3]_{12}$

m of the x boxes are of the type $[1\ ;\ 2]_{ii+1i+2i+3}$

n of the y boxes are of the type $[1\ ;\ 2]_{ii+1}$

So totally in the first row we have

k_1-1-y elements of the type $[1\ ;\ i+1]_{12}$

n elements of the type $[1\ ;\ 2]_{ii+1}$

y-n elements of the type $[1,i\ ;\ ]$

m elements of the type $[1\ ;\ 2]_{ii+1i+2i+3}$

x-m elements of the type $[1,i,i+2\ ;\ ]$

s elements of the type $[1\ ;\ 2]_{ii+1i+2i+3}$

r-s elements of the type $[1,i+2\ ;\ i+1,i+3]_{12}$

k_2-r-x elements of the type $[1,i+2\ ;\ i+1,i+3]_{12}'$

i.e.,

k_1-1-y elements are of the type $[1\ ;\ i+1]_{12}$

n elements are of the type $[1\ ;\ 2]_{ii+1}$

y-n elements are of the type $[1,i\ ;\ ]$

m+s elements are of the type $[1\ ;\ 2]_{ii+1i+2i+3}$

$x-n$ elements are of the type $[i,i,i+2 \; ; \;]$

k_2-x-s elements are of the type $[i,i+2 \; ; \; i+1,i+3]_{12}$

So, we get

$$\sum_{n=0}^{y} {}^{y}C_n \sum_{n=0}^{x} {}^{x}C_n \, A^{(2,\cdot)}_{k_1-1-y,\,k_2-x-s}(r-s) \, A^{(2,\cdot)}_{1,\,n+s}(s) \tag{5.60}$$

Choices.

Case 2

If at least one element in a box in the first row is of the form $[\;\; ; \; 2, i+1]$. Then, all the y elements in the boxes should be of the type $[1 \; ; \; 2]_{ii+1}$ and all x elements in the boxes should be of the type

$$[1 \; ; \; 2]_{ii+1i+2i+3}$$

We have

k_2-x-r elements of the type $[i,i+2 \; ; \; i+1,i+3]_{12}$

and s elements of the type $[1 \; ; \; 2]_{ii+1i+2i+3}$

Let t of the k_1-1-y be of the type $[i \; ; \; i+1]_{12}$

and L of the $r-s$ be of the type $[i,i+2 \; ; \; i+1,i+3]_{12}$

So in all in the first row we have

y elements of the type $[1 \; ; \; 2]_{ii+1}$

$k_1-1-y-t$ elements of the type $[\;\; ; \; 2,i+1]$

t elements of the type $[i \; ; \; i+1]_{12}$

x elements of the type $[1 \; ; \; 2]_{ii+1i+2i+3}$

k_2-x-r elements of the type $[i,i+2 \; ; \; i+1,i+3]_{12}$

s elements of the type $[i\ ;\ 2]_{ii+1i+2i+3}$

L elements of the type $[i,i+2\ ;\ i+1,i+3]_{12}$

r-s-L elements ofthe type $[\ ;\ 2,\ i+1,i+3]$

That is,

y elements of the type $[i\ ;\ 2]_{ii+1}$

k_1-1-y-t elements of the type $[\ ;\ 2,i+1]$

t elements of the type $[i\ ;\ i+1]_{12}$

x+s elements of the type $[i\ ;\ 2]_{ii+1i+2i+3}$

k_2-x-r+L elements of the type $[i,i+2\ ;\ i+1,\ i+3]_{12}$

r-s-L elements of the type $[\ ;\ 2,i+1,i+3]$

So finally we get

$$\sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L \ A^{(2,4)}_{y,x+s(s)} \ A^{(2,4)}_{t,k_2-x-r+L(r-s)} \qquad (5.61)$$

Choices for the remaining rows.

From (5.60) and (5.61) for a particular y,x and s the contribution is

$$\sum_{n=0}^{y} {}^{y}C_n \sum_{n=0}^{x} {}^{x}C_n \ A^{(2,4)}_{k_1-1-y,k_2-x-s(r-s)} \ A^{(2,4)}_{n,n+s(s)}$$

$$+ \sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L \ A^{(2,4)}_{y,x+s(s)} \ A^{(2,4)}_{t,k_2-x-r+L(r-s)}$$

The term

$A^{(2,4)}_{k_1-1-y,k_2-x-r(r-s)} \ A^{(2,4)}_{y,x+s(s)}$ is counted twice, so we can subtract it from the summation.

Now s varies from 0 to r

x varies from 0 to k_2-r

y varies from 0 to k_1-1

So the total contribution to $A^{(2,4)}_{k_1,k_2(r)}$ is

$$= \sum_{y=0}^{k_1-1} {}^{k_1-1}C_y \sum_{x=0}^{k_2-r} {}^{k_2-r}C_x \sum_{s=0}^{r} {}^{r}C_s$$

$$\left\{ A^{(2,4)}_{k_1-1-y,k_2-x-r(r-s)} \sum_{n=0}^{y} {}^{y}C_n \sum_{m=0}^{x} {}^{x}C_m \, A^{(2,4)}_{n,m+s(s)} \right.$$

$$+ A^{(2,4)}_{y,x+s(s)} \sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L \, A^{(2,4)}_{k_1-1-y,k_2-x-s+L(r-s)}$$

$$\left. - A^{(2,4)}_{k_1-1-y,k_2-x-r(r-s)} \, A^{(2,4)}_{y,x+s(s)} \right\} \tag{5.62}$$

And now substituting

$$a(x,y,s) = A^{(2,4)}_{k_1-1-y,k_2-x-r(r-s)}$$

$$b(m.n,s) = \sum_{n=0}^{y} {}^{y}C_n \sum_{m=0}^{x} {}^{x}C_m \, A^{(2,4)}_{n,m+s(s)}$$

$$c(y,x,s) = A^{(2,4)}_{y,r+s(s)}$$

$$d(t,L,s) = \sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L \, A^{(2,4)}_{k_1-1-y,k_2-x-s+L(r-s)}$$

So, (5.62) takes the form

$$A^{(2,4)}_{k_1,k_2}(r) = \sum_{y=0}^{k_1-1} {}^{k_1-1}C_y \sum_{x=0}^{k_2-r} {}^{k_2-r}C_x \sum_{s=0}^{r} {}^{r}C_s$$

$$\left\{a(x,y,s)\ b(m,n,s) + c(x,y,s)\ d(t,L,s) - a(x,y,s)\ c(x,y,s)\right\} \tag{5.63}$$

Total contribution from k_1 cycles of length 2 and k_2 cycles of length 4 is

$$2^{k_1+k_2} \sum_{r=0}^{k_2} {}^{k_2}C_r \ A^{(2,4)}_{k_1,k_2}(r) \tag{5.64}$$

Hence the lemma 5.4.

Lemma 5.5

Cycle of length 1 does not contribute anything independently with a cycle of length 2 or a cycle of length 4.

Proof

This can be easily proved. By lemma 5.5 if the permutation

$$\beta = (1^{j_1}\ 2^{j_2}\ 4^{j_4}\ \ldots .)\ j_1 = 0 \text{ or } 1.$$

Then the contribution to s.c.t. from β is same as from the permutation

$$\alpha = (2^{j_2}\ 4^{j_4}\ 6^{j_6}\ \ldots .\ )$$

From lemma 5.2. The contribution is independent of the lengths of the cycles, if the cycle is of length greater than 2.

Hence contribution to s.c.t. digraphs from

$\alpha = (2^{j_2} 4^{j_4} 6^{j_6} \ldots\ldots)$ is same as from the permutation

$$\gamma = (2^{k_1} 4^{k_2})$$

where

$$k_1 = j_2 \quad \text{and} \quad k_2 = j_4 + j_6 + j_8 + \ldots\ldots$$

Theorem 1

The number of selfcomplementary transitive digraph of order p is

$$\frac{1}{p!} \sum_{\alpha \in S_p} T(\alpha) \tag{5.65}$$

$$T(\alpha) = 2^{k_1} \sum_{r=0}^{k_1-1} {}^{k_1-1}C_r \left(A^{(2)}_{k_1-1-r}\, \varphi_1(n) + A^{(2)}_r\, \varphi_2(m) - A^{(2)}_{k_1-1-r}\, A^{(2)}_r \right) \tag{5.66}$$

where $\varphi_1(n) = \sum_{n=0}^{r} {}^{r}C_n\, A^{(2)}_n$

$$\varphi_2(m) = \sum_{m=0}^{k_1-1-r} {}^{k_1-1-r}C_m\, A^{(2)}_m \quad \text{for } \alpha = 2^{k_1} \tag{5.67}$$

$$T(\alpha) = 2^{k_2} \sum_{r=0}^{k_2} {}^{k_2}C_r \sum_{n=0}^{r} {}^{r}C_n \sum_{x=0}^{k_2-1-r} {}^{k_2-1-r}C_x \sum_{m=0}^{x} {}^{x}C_m\, \varphi(m,n,x) \tag{5.68}$$

$$\text{where } \varphi(m,x,n) = A_{k_2-1-x-n(r-n)}\, A^{(4)}_{m+n(n)} \tag{5.69}$$

and
where

$$\alpha = (4^{j_4}\ 6^{j_6}\ \ldots\ldots) \text{ and } k_2 = \sum_{i=4,6\ldots} j_i$$

and if $k_1 \neq 0$ and $k_2 \neq 0$.

$$T(\alpha) = 2^{k_1+k_2} \sum_{r=0}^{k_2} {}^{k_2}C_r\ A^{(2,4)}_{k_1,k_2}(r)$$

where

$$A^{(2,4)}_{k_1,k_2}(r) = \sum_{y=0}^{k_1-1} {}^{k_1-1}C_y \sum_{x=0}^{k_2-r} {}^{k_2-r}C_x \sum_{s=0}^{r} {}^{r}C_s$$

$$\left\{a(x,y,s)\ b(m,n,s) + c(x,y,s)\ d(L,t,s) - a(x,y,s)\ c(x,y,s)\right\}$$

(7(5.70)

where

$$a(x,y,s) = A^{(2,4)}_{k_1-1-y,k_2-x-r}(r-s)$$

$$b(m,n,s) = \sum_{n=0}^{y} {}^{y}C_n \sum_{m=0}^{x} {}^{x}C_m\ A^{(2,4)}_{n,m+s}(s)$$

$$c(x,y,s) = A^{(2,4)}_{y,x+s}(s)$$

$$d(t,L,s) = \sum_{t=0}^{k_1-1-y} {}^{k_1-1-y}C_t \sum_{L=0}^{r-s} {}^{r-s}C_L\ A^{(2,4)}_{k_1-1-y,k_2-x-s+L}(r-s)$$

Proof

Proof follows from lemmas 5.1, 5.2, 5.3 and 5.4, by Substituting

$$k_1 = j_2 \text{ and } k_2 = j_4 + j_6 + j_8 \ldots\ldots$$

When $\alpha = (1^{j_1}\, 2^{j_2}\, 4^{j_4} \ldots\ldots)$

The numbers of selfcomplementary transitive digraphs upto p = 17 are calculated and are listed below.

p	1	2	3	[illegible]	5	6	7	8	9	10	11
s.c.t. digraphs	1	1	1	3	3	7	7	17	17	41	41

p	12	13	14	15	16	17
s.c.t. digraphs	99	99	239	239	577	577

Diagram for s.c.t. digraphs of order 5 is as in Fig. 5.1.

By using the same procedure, one can show that the number of s.c.t. oriented graph is unity for any order of p. This result is proved using a different argument in [25]. We can easily show that the degree sequence of s.c.t. oriented graph will be

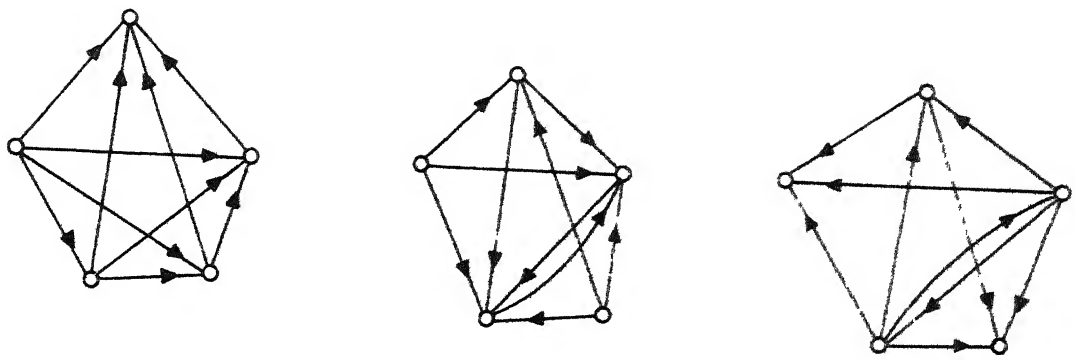
$$(p-1,0),(p-2,1)\quad (p-3,2)\ \ldots\ ,\ (1,p-2),\ (0,p-1)$$

So the automorphism group of a s.c.t. oriented graph consists of only identity permutation. So, the number of ways of labelling s.c.t. oriented graph of order p is p!.

5.2 Eulerian Transitive Digraphs

Theorem 5.2

There is only one transitive eulerian digraph of order p.



Self-complementary transitive digraphs of order 5

FIG. 5.1

Proof

An eulerian digraph is a weakly connected isograph. The degree sequence of an isograph of order p is

$$(k_1,k_1),\ (k_2,k_2)\ ,\ \ldots\ldots\ (k_p,k_p) \qquad 0 < k_1,k_2,\ \ldots\ k_p \leqslant p-1$$

We first show that the degree sequence of an eulerian transitive digraph is

$$(k,k),\ (k,k),\ (k,k),\ \ldots\ldots\ldots\ldots\ (k,k) \qquad 0 < k \leqslant p-1.$$

Consider two adjacent vertices u and v with degree (k_1,k_1) and (k_2,k_2) where $k_1 \neq k_2$.

Let $k_1 < k_2$, and suppose that uv is an edge, and vw_i are edges for $i = 1,2,3,\ \ldots\ k_2$. For a transitive digraph we should have edges uw_i $\quad i = 1,2,3,\ \ldots\ k_2$, since uv is an edge. This implies that u has at least k_2 out degree. But $k_1 < k_2$ which is a contradiction. Similarly we will have a contradiction if we assume the edge vu exists. So u and v both have degree (k,k).

Now since eulerian digraph is weakly connected, we can continue the process to all vertices.

So we have the degree sequence of an eulerian transitive digraph as

$$(k,k),\ (k,k),\ (k,k),\ \ldots\ldots\ (k,k), \text{ where } 0 < k \leqslant p-1.$$

Now we show that the only value that k can take is $k = p-1$. Suppose that it is not. Then there exist at least two vertices

u and v such that uv is an edge but not vu. Consider the edge uv and the edges vw_i for i = 1,2,3, ... k. Since the digraph is transitive, uw_i for i = 1,2,3, ... k should also be edges of the digraph. Therefore the outdegree of the vertex u is at least k+1 which is a contradiction. So the degree sequence of an eulerian transitive digraph is

$$(p-1,\ p-1),\ (p-1,\ p-1),\ (p-1,p-1),\ \ldots,\ (p-1,\ p-1).$$

which is a complete digraph.

5.3 Hamiltonian Transitive Digraph

Theorem 5.3

There is only one transitive hamiltonian digraph of order p.

Proof

A hamiltonian digraph contains a spanning directed cycle. Let C be a spanning directed cycle of a digraph D. Denote its vertices by $v_1, v_2, v_3, v_4, \ldots\ v_p$. Let

$C = v_1, v_1v_2, v_2, v_2v_3, v_3, v_3v_4, \ldots, v_{p-1}v_p, v_p, v_pv_1$.

Now we will prove that for any two vertices v_i and v_j both v_iv_j and v_jv_i are edges if we impose the transitive property. Start with v_1. We have v_1v_2 and v_2v_3, since D is transitive v_1v_3 also an edge in D. We have v_1v_3 and v_3v_4 D is transitive implies that v_1v_4 is also an edge in D. Proceeding in this way we have all the edges

$$v_1 v_k \quad k = 2,3,4,5, \ldots p.$$

Similarly we can prove that

$v_2 v_k$ for $k = 1,3,4,5, \ldots p$ are edges of D.

$v_3 v_k$ for $k = 1,2,4,5, \ldots p$ are edges of D, etc.

Therefore, $v_i v_j$ and $v_j v_i$ are edges. That is, D is a complete digraph. Hence a hamiltonian transitive digraph of order p is a complete digraph of order p.

REFERENCES

1. Baken, H.; Dewdeny, A.K. and Szilard, A.L., Generating nine point graphs, Math Computation, 28, 833-838 (197-).

2. Berge, Graphs and Hypergraphs, North Holland Amesterdum (1973).

3. Burnside, W., 'Theory of Groups of Finite Order' 2nd ed., p 191 Theorem VII. Cambridge Univ.Press, London 1911, Printed by Dover, New York (1955).

4. Cayley, A., On the theory of the analytical forms called trees, Philos. Mag., 13, 19-30 (1857).

5. Davis, R.L., The number of structures of finite relations, proc. Amer. Math. Soc., 4, 486-495 (1953).

6. DeBruijn, N.C., Generalization of Polya's Fundamental theorem in Enumerative Combinatorial Analysis, Indag. Math., 21 (Nov.2), 59-69 (1959).

7. DeBruijn, N.G., Enumerative Combinatorial Problems Concerning Structures, Nieuw. Arch. Wisk., 3, 142-161 (1963).

8. DeBruijn, N.G., Polya's theory of counting in 'Applied Combinatorial Mathematics' (E.F. Beckenback, ed) Wiley, New York pp 144-18? (1964).

9. DeBruijn, N.C. and Klarner, D.A., Enumeration of generalized graphs, Indag. Math., 31, 1-9 (1969).

10. Evans, J.W.; Harary, F. and Lynn, M.S., On the computer enumeration of finite topologies, Comm. ACM, 10, 295-298 (1967).

11. Frucht, Roberto and Harary, F., Selfcomplementary generalized orbits of a permutation group, Canad. Math. Bull., 17, 203-208 (1974).

12. Foulkes, H.O., On Redfield's group reduction functions, Canad. J. Math., 15, 272-284 (1963).

13. Foulkes, H.O., On Redfield's range - correspondence, Canad. J. Math., 18, 1060-1071 (1966).

14. Harrison, M.A., A census of finite automata, Canad. J. Math., 17, 100-113 (1965).

15. Harrison, M.A., 'Introduction to Switching theory and automata', McGraw Hill, New York (1965).

16. Harary, F., The number of linear, directed, rooted, and connected graphs, Trans. Amer. Math. Sol., 78, 445-463 (1965 (1955).

17. Harary, F., Unsolved problems in the enumeration of graphs, Publ. Math. Inst. Math., 4, (128-135 (1962).

18. Harary, F., Combinatorial problems in graphical enumeration 'Applied Combinatorial Mathematics', (E.F. Beckenbach, ed), Wiley New York, pp 185-200 (1965).

19. Harary, F. and Palmer, E.M., Enumeration of selfconverse digraphs, Mathematika, 13, 151-157 (1966).

20. Harary, F. and Palmer, E.M., Enumeration of mixed graphs, Proc. Amer. Math. Soc., 17, 682-687 (1966).

21. Harary, F. and Palmer, E.M., Enumeration of locally restricted digraphs, Canad. J. Math., 18, 853-860 (1966).

22. Harary, F. and Palmer, E.M., The enumeration methods of Redfield, Amer. J. Math., 89, 373-384 (1967).

23. Harary, F., Graphical enumeration problems. Chapter 1 in 'Graph theory and theoretical physics' (F. Harary, ed), Academic Press, London, pp 1-41 (1967).

24. Harary, F. and Palmer, E.M., 'Graphical enumeration', Academic Press, New York (1973).

25. Harary; Norman, R.Z. and Cartwright, Structural models, 'An Introduction to the one of directed graphs', Wiley New York (1965).

26. Jackson, D.M., 'Mathematical methods in Enumerative Combinatorial Theory', Academic Press, New York (1977).

27. Klarner, D.A., Enumeration involving sums over compositions, Ph.D. thesis, University of Alberta (1966).

28. Klarner, D.A. and DeBruijn, N.C., 'Pattern Enumeration' (to appear).

29. Krishnamurthy, V., Counting of finite topologies and a dissection of stirling numbers of second kind, Bull. Austral. Math. Soc., 12, 111-124 (1975).

30. Morris, P.A., Selfcomplementary graphs and digraphs, Math. Computation, 27, 216-217 (1973).

31. Parthasarathy, K.R., Enumeration of graphs with a given partition, Canad. J. Math., 20, 40-47 (1968).

32. Polya, G., Kombinatorische Anzahlbestimmungen fur Gruppen, Graphen Und Chemische Verbindungen, Acta. Math., 68, 145-254 (1937).

33. Polya, G., 'How to solve it', Princeton University Press, Princeton (1965).

34. Read, R.C., The enumeration of locally restricted graphs Ind II, J. Lon. Math. Soc. 34, (1959) 417-436 ; 35, (1960) 334-351.

35. Read, R.C., The number of k-colored graphs on labelled nodes, Canad. J. Math., 12, 409-413 (1960).

36. Read, R.C., Euler graphs on labelled nodes, Canad. J. Math., 14, 482-486 (1962).

37. Read, R.C., On the number of selfcomplementary graphs and digraphs, J. London Math. Soc., 38, 99-104 (1963).

38. Read, R.C., The use of S-functions in combinatory analysis, Canad. J. Math., 20, 808-841 (1968).

39. Read, R.C., Teaching graph theory to computer, 'Recent progress in combinatorics, Academic press, New York (1968).

40. Redfield, J.H., The theory of group-reduced distributions, Amer. J. Math., 49, 433-455 (1927).

41. Robinson, R.W., Enumeration of euler graphs, in 'proof technique in graph theory', (F. Harary, ed), pp 147-153, Academic Press, New York (1969).

42. Robinson, R.W., Enumeration of non separable graphs, J.C. Theory, 9, 327-356 (1970).

43. Sharp, Henry, Jr., Enumeration of transitive step type relations, Acta. Math. Acad. Sci. Hung., 22 (1972).

44. Sharp, Jr., Enumeration of vacuously transitive relations, Discrete. Math., 4, 185-196 (1973).

45. Shapankumar Das, On enumeration of finite maximal connected topologies, J. Combinatorial theory, B, 15, 184-199 (1973).

46. Sheehan, J., On polya's theorem, Canad. J. Math., 19, 792-799 (1967).

47. Sheehan, J., The number of graphs with a given automorphism group, Canad. J. Math., 20, 1068-1076 (1968).

48. Sridharan, M.R. and Parthasarathy, K.R., Enumeration of selfcomplementary graphs and digraphs, J. Math. Phy. Sci., 14 410 (1969).

49. Sridharan, M.R., Selfcomplementary and selfconverse oriented graphs, Indag. Math., 32, 441-447 (1970).

50. Sridharan, M.R. and Parthasarathy, K.R., On structure enumeration theory, Indag. Math., 33, (No.4), 327-339 (1971).

51. Sridharan, M.R. and Parthasarathy, K.R., Isographs and oriented Isographs, J. Combi. Theory, 13 (No.2), (1972).

52. Sridharan, M.R., Mixed selfcomplementary and selfconverse digraphs, Discrete. Math., 14, 373-376 (1976).

53. Sridharan, M.R. and Parthasarathy, K.R., Enumeration of graphs and digraphs with local restrictions, J. Math. Phy. Sci., 5, 483-490 (1977).

54. White, D.E., Redfield's theorems and multilinear algebra, Canad. J. Math., 3, 704-714 (1975).

55. White, D.E., Classifying patterns by automorphism group, Discrete, Math., 13, 277-295 (1975).

56. Williamson, S.G., Polya's counting theorem and a class of tensor identities, J. London. Math. Soc., 3, 411-421 (1971).

57. Williamson, S.G., Isomorph rejection and theorem of deBruijn, SIAM J. Compute., 44-59 (1973).